Separation principle

For any set A and any definable property P, there is a set containing precisely those elements of A for which the property P holds.

$$\{x \in A \mid \mathrm{P}(x)\}$$

$$a \in \{x \in A \mid P(x)\} \iff (a \in A) \wedge P(a)$$

NB:

$$\{x \in A \mid P(x)\} \subseteq \{y \in B \mid Q(y)\}$$

is equivalent to

$$\forall z. \left[(z \in A) \wedge P(z)\right] \Rightarrow \left[(z \in B) \wedge Q(z)\right]$$

Russell's paradox

Let

$$U =^{def} \{x \mid R(x)\} \text{ for } R(x) =^{def} (x \notin x)$$

Then as

$$U \in \{x \mid R(x)\} \Longleftrightarrow R(U)$$

it follows that

$$U \in U \Longleftrightarrow U \notin U$$

Empty set

- The theory provides an empty set, with no elements
- This is,

$$\emptyset =^{def} \{ x \in A \mid \underline{false} \}$$

- Indeed, $a \in \emptyset \Leftrightarrow \underline{false}$

That is, $\forall a.\ a \notin \emptyset$

NB: for all sets A and B,

$$\{ x \in A \mid \underline{\text{false}} \} = \{ y \in B \mid \underline{\text{false}} \}$$

NB: for all sets A,

$$\emptyset \subseteq A$$

Cardinality

The *cardinality* of a set specifies its size. If this is a natural number, then the set is said to be *finite*.

Typical notations for the cardinality of a set S are $\#S$ or $|S|$.

Example:

$$\#\emptyset = 0$$

FINITE SETS

The finite sets are those with cardinality a natural number

Example: For $n \in \mathbb{N}$,

$$[n] =_{\text{def}} \{ x \in \mathbb{N} \mid x < n \}$$

is finite of cardinality n.