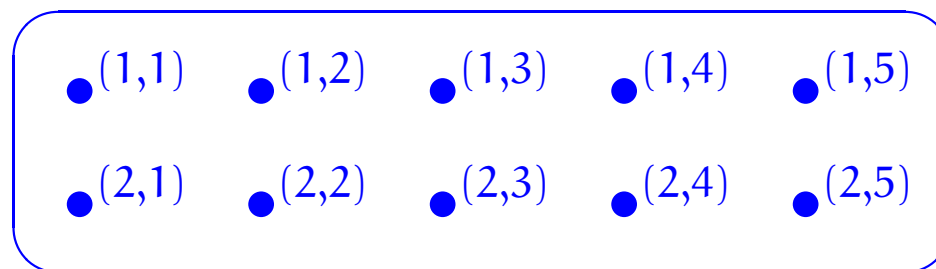


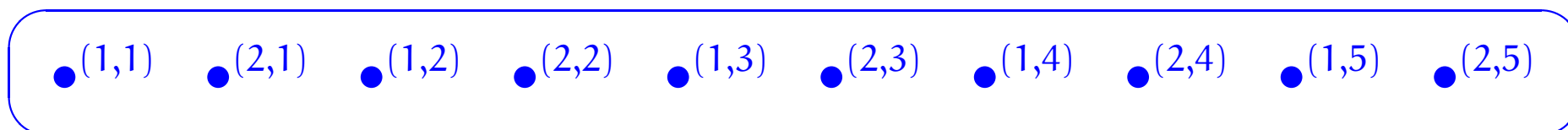
Sets

Abstract sets

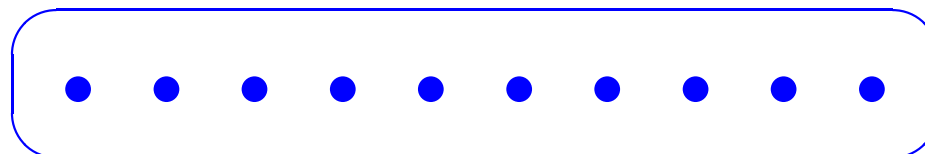
It has been said that a set is like a mental “bag of dots”, except of course that the bag has no shape; thus,



may be a convenient way of picturing a certain set for some considerations, but what is apparently the same set may be pictured as



or even simply as



for other considerations.

Naive Set Theory

We are not going to be formally studying Set Theory here; rather, we will be *naively* looking at ubiquitous structures that are available within it.

Set membership

We write \in for the membership predicate;
so that

$x \in A$ stands for x is an element of A

We further write

$x \notin A$ for $\neg(x \in A)$

Example: $0 \in \{0, 1\}$, $1 \notin \{0\}$

Extensionality axiom

Two sets are equal if they have the same elements.

Thus,

$$\forall \text{ sets } A, B. A = B \iff (\forall x. x \in A \iff x \in B) .$$

NB: $1 \in \{0, 1\} \wedge 1 \notin \{0\}$

Example:

$\{0\} \neq \{0, 1\} = \{1, 0\} \neq \{2\} = \{2, 2\}$

NB: $x \in \{2\} \iff (x=2) \iff x \in \{2, 2\}$

Proposition For $b, c \in \mathbb{R}$, let

$$A \stackrel{\text{def}}{=} \{ x \in \mathbb{C} \mid x^2 - 2bx + c = 0 \}$$

$$B \stackrel{\text{def}}{=} \{ b + \sqrt{b^2 - c}, b - \sqrt{b^2 - c} \}$$

$$C \stackrel{\text{def}}{=} \{ b \}$$

Then,

$$(1) A = B,$$

and

$$(2) b^2 = c \iff B = C.$$

PROOF IDEA:

$$(1) \forall x \in \mathbb{C}.$$

$$x^2 - 2bx + c = 0$$

$$\Leftrightarrow \left(x = b + \sqrt{b^2 - c} \quad \vee \quad x = b - \sqrt{b^2 - c} \right).$$

$$(2) b^2 = c$$

$$\Rightarrow \left[\forall x. \left(x = b + \sqrt{b^2 - c} \quad \vee \quad x = b - \sqrt{b^2 - c} \right) \right. \\ \left. \Leftrightarrow x = b \right]$$



Subsets and supersets

We write

$$A \subseteq B$$

for

$$\forall x. x \in A \Rightarrow x \in B.$$

Example: $\{0\} \subseteq \{0, 1\}.$

Lemma 83

1. *Reflexivity.*

For all sets A , $A \subseteq A$.

2. *Transitivity.*

For all sets A, B, C , $(A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$.

3. *Antisymmetry.*

For all sets A, B , $(A \subseteq B \wedge B \subseteq A) \implies A = B$.

Proper subsets

We let $A \subset B$ stand for $A \subseteq B \wedge A \neq B$

Hence

$$\text{iff } A \subset B \\ (\forall x. x \in A \Rightarrow x \in B) \wedge (\exists y. y \notin A \wedge y \in B)$$

Example: $\{0\} \subset \{0, 1\}$