Sets

Abstract sets

It has been said that a set is like a mental "bag of dots", except of course that the bag has no shape; thus,

$$\boxed{\begin{array}{ccccc} \bullet^{(1,1)} & \bullet^{(1,2)} & \bullet^{(1,3)} & \bullet^{(1,4)} & \bullet^{(1,5)} \\ \bullet^{(2,1)} & \bullet^{(2,2)} & \bullet^{(2,3)} & \bullet^{(2,4)} & \bullet^{(2,5)} \end{array}}$$

may be a convenient way of picturing a certain set for some considerations, but what is apparently the same set may be pictured as

$$\boxed{\begin{array}{cccccccccc} \bullet^{(1,1)} & \bullet^{(2,1)} & \bullet^{(1,2)} & \bullet^{(2,2)} & \bullet^{(1,3)} & \bullet^{(2,3)} & \bullet^{(1,4)} & \bullet^{(2,4)} & \bullet^{(1,5)} & \bullet^{(2,5)} \end{array}}$$

or even simply as

$$\boxed{\begin{array}{cccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}}$$

for other considerations.

Naive Set Theory

We are not going to be formally studying Set Theory here; rather, we will be *naively* looking at ubiquitous structures that are available within it.

Set membership

We write $\in$ for the membership predicate; so that

$x \in A$ stands for x is an element of A

We further write

$x \notin A$ for $\neg(x \in A)$

Example: $0 \in \{0, 1\}$, $1 \notin \{0\}$

Extensionality axiom

Two sets are equal if they have the same elements.

Thus,

$$\forall \text{ sets } A, B.\ A = B \iff (\forall x.\, x \in A \iff x \in B) \quad .$$

NB: $1 \in \{0, 1\} \wedge 1 \notin \{0\}$

Example:

$$\{0\} \neq \{0, 1\} = \{1, 0\} \neq \{2\} = \{2, 2\}$$

NB: $x \in \{2\} \iff (x = 2) \iff x \in \{2, 2\}$

Proposition For $b, c \in \mathbb{R}$, let

$$A \overset{\text{def}}{=} \{ x \in \mathbb{C} \mid x^2 - 2bx + c = 0 \}$$

$$B \overset{\text{def}}{=} \{ b + \sqrt{b^2 - c}, \ b - \sqrt{b^2 - c} \}$$

$$C \overset{\text{def}}{=} \{ b \}$$

Then,

(1) $A = B$,

and

(2) $b^2 = c \iff B = C$.

PROOF IDEA:

(1) $\forall x \in \mathbb{C}.$

$$x^2 - 2bx + c = 0$$

$$\iff \left(x = b + \sqrt{b^2 - c} \lor x = b - \sqrt{b^2 - c} \right).$$

(2) $b^2 = c$

$$\Rightarrow \left[\forall x. \left(x = b + \sqrt{b^2 - c} \lor x = b - \sqrt{b^2 - c} \right) \iff x = b \right]$$

$\boxtimes$

Subsets and supersets

We write

$$A \subseteq B$$

for

$$\forall x .\, x \in A \Rightarrow x \in B .$$

Example: $\{0\} \subseteq \{0,1\}$.

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1. *Reflexivity.*

   For all sets A, $A \subseteq A$.

2. *Transitivity.*

   For all sets A, B, C, $(A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$.

3. *Antisymmetry.*

   For all sets A, B, $(A \subseteq B \wedge B \subseteq A) \implies A = B$.

Proper subsets

We let $A \subset B$ stand for $A \subseteq B \wedge A \neq B$

Hence

$A \subset B$

Iff $(\forall x.\, x \in A \Rightarrow x \in B) \wedge (\exists y.\, y \notin A \wedge y \in B)$

Example: $\{0\} \subset \{0, 1\}$