Natural Numbers and mathematical induction

We have mentioned in passing that the natural numbers are generated from zero by succesive increments. This is in fact the defining property of the set of natural numbers, and endows it with a very important and powerful reasoning principle, that of *Mathematical Induction*, for establishing universal properties of natural numbers.

Principle of Induction

Let $P(m)$ be a statement for m ranging over the set of natural numbers $\mathbb{N}$.

If

- the statement $P(0)$ holds, and
- the statement

  $$\forall n \in \mathbb{N}. \; \big(\, P(n) \implies P(n+1) \, \big)$$

  also holds

then

- the statement

  $$\forall m \in \mathbb{N}. \, P(m)$$

  holds.

A template for induction proofs:

1. State that the proof uses induction.
2. Define an appropriate property $P(m)$ for m ranging over the set of natural numbers. This is called the *induction hypothesis*.
3. Prove that $P(0)$ is true. This is called the *base case*.
4. Prove that $P(n) \implies P(n+1)$ for every natural number n. This is called the *inductive step*.
5. Invoke the principle of mathematical induction to conclude that $P(m)$ is true for all natural numbers m.

NB Always be sure to explicitly label the *induction hypothesis*, the *base case*, and the *inductive step*.

Binomial Theorem

Theorem 29 *For all* $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} \cdot y^k .$$

PROOF: We prove

$\forall n \in \mathbb{N}.\, P(n)$

where $P(n) =_{\text{def}} \left[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right]$

by the Principle of Mathematical Induction.

Base case: We need show the property for 0; explicitly

$$(x+y)^0 \stackrel{?}{=} \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k$$

- $(x+y)^0 = 1$
- $\sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k = \binom{0}{0} . x^0 y^0 = 1$

Hence, $P(0)$ holds.

Inductive step Let n be a natural number.

Assume the Induction Hypothesis

$$(IH) \quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

RTP

$$(x+y)^{n+1} \overset{?}{=} \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k$$

$$(x+y)^{n+1} = (x+y) \cdot (x+y)^n$$

$$\overset{\text{by (IH)}}{=} (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(x+y)^{n+1} = x\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k} + y\cdot\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^{k} + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}$$

$$= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^{k} + \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^{k} + y^{n+1}$$

$$= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]x^{n+1-k}y^{k}$$

by Lemma

$$\left[\binom{n}{k}+\binom{n}{k-1}\right] = \binom{n+1}{k}$$

$$= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^{n+1-k}y^{k}$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^{k} \qquad \boxtimes$$