Negation

Negations are statements of the form

$$\boxed{\text{not } P}$$

or, in other words,

$$\boxed{P \text{ is not the case}}$$

or

$$\boxed{P \text{ is absurd}}$$

or

$$\boxed{P \text{ leads to contradiction}}$$

or, in symbols,

$$\boxed{\neg P}$$

A first proof strategy for negated goals and assumptions:

If possible, reexpress the negation in an *equivalent* form and use instead this other statement.

Logical equivalences

$$\neg(P \Longrightarrow Q) \iff P \wedge \neg Q$$

$$\neg(P \iff Q) \iff P \iff \neg Q$$

$$\neg(\forall x.\, P(x)) \iff \exists x.\, \neg P(x)$$

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$

$$\neg(\exists x.\, P(x)) \iff \forall x.\, \neg P(x)$$

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

$$\neg(\neg P) \iff P$$

$$\neg P \iff (P \Rightarrow \mathbf{false})$$

Theorem 37 *For all statements* P *and* Q,

$$(P \implies Q) \implies (\neg Q \implies \neg P) .$$

PROOF: Let P and Q be statements.

Assume $P \Rightarrow Q$ (1)

RTP $\neg Q \Rightarrow \neg P$

Assume $\neg Q \iff (Q \Rightarrow \text{false})$ (2)

RTP $\neg P \iff (P \Rightarrow \text{false})$ (3)

From (1) and (2) we have (3) as required

$\boxtimes$

<u>NB:</u>

Amongst the equivalences for negation we have postulated the somewhat controversial

$$\neg\neg P \iff P$$

which is <u>classically</u> accepted.

In this light, to prove

$$P$$

one may equivalently prove

$$(\neg P) \Rightarrow \underline{false}.$$

That is,

assuming $\neg P$ leads to contradiction

This technique is know as

PROOF BY CONTRADICTION

Proof by contradiction

The strategy for proof by contradiction:

To prove a goal P by contradiction is to prove the equivalent statement $\neg\mathrm{P} \implies \mathbf{false}$

Proof pattern:

In order to prove

$$\mathrm{P}$$

1. Write: We use proof by contradiction. So, suppose P is false.
2. Deduce a logical contradiction.
3. Write: This is a contradiction. Therefore, P must be true.

Scratch work:

Before using the strategy

Assumptions	Goal
	P
$\vdots$	

After using the strategy

Assumptions	Goal
	contradiction
$\vdots$	
$\neg P$	

Theorem 39 *For all statements* P *and* Q,

$$(\neg \mathrm{Q} \implies \neg \mathrm{P}) \implies (\mathrm{P} \implies \mathrm{Q}) \ .$$

PROOF: Let P and Q be statements.

Assume $\neg Q \Rightarrow \neg P$ (1)

RTP $P \Rightarrow Q$

Assume P (4)

RTP Q

Assume $\neg Q$ (by contradiction) (2)

RTP false.

(3)

From (1) and (2), we deduce $\neg P \iff (P \Rightarrow \text{false})$

From (3) and (4), false follows as required

$\boxtimes$

NB:

We have proved

$$(P \Rightarrow Q) \Longleftrightarrow (\neg Q \Rightarrow \neg P)$$

which, in fact, we have already used as the technique of

PROOF BY THE CONTRAPOSITIVE

Lemma 41 *A positive real number* x *is rational iff*

$$\exists \textit{ positive integers } m, n : \qquad (\dagger)$$
$$x = m/n \wedge \neg(\exists \textit{ prime } p : p \mid m \wedge p \mid n)$$

PROOF: Let x be a positive real number.

$(\dagger) \Rightarrow x$ is rational:

Assume $(\dagger)$. Then $x = m/n$ for some pos. int m and n; Therefore rational.

x rational $\Rightarrow (\dagger)$

$(\dagger)\quad \exists$ pos. int. $m, n.$

$$x = m/n \wedge \neg(\exists \text{ prime } p.\ p \mid m \wedge p \mid n)$$

$\neg(\dagger) \Leftrightarrow \forall$ pos. int. $m, n.$

$$\neg\left[x = m/n \wedge \neg(\exists \text{ prime } p.\ p \mid m \wedge p \mid n)\right]$$

$\Leftrightarrow \forall$ pos int $m, n.$

$$\neg(x = m/n) \vee (\exists \text{ prime } p.\ p \mid m \wedge p \mid n)$$

$\Leftrightarrow \forall$ pos. int. $m, n.$

$$(x = m/n \Rightarrow \exists \text{ prime } p.\ p \mid m \wedge p \mid n)$$

- **x rational $\Rightarrow$ (†)**

Assume (1) $x = m_0/n_0$ for some pos. int. m_0, n_0

Assume $(*)$ $\forall$ pos. int. m, n:

$$x = m/n \Rightarrow \exists \text{ prime } p : p \mid m \wedge p \mid n$$

We need deduce a contradiction.

Instantiating $(*)$, we have:

(2) $x = m_0/n_0 \Rightarrow \exists$ prime $p : p \mid m_0 \wedge p \mid n_0$

From (1) and (2), we have

(3) $\exists$ prime p. $p \mid m_0 \wedge p \mid n_0$

There is a prime p_0 such that $p_0 | m_0 \wedge p_0 | n_0$.
That is, there are positive integers m_1 and n_1 such that

$$m_0 = p_0 \cdot m_1 \quad \text{and} \quad n_0 = p_0 \cdot n_1$$

Also

$$x = m_1 / n_1$$

Instantiating (*)

$$x = m_1 / n_1 \Rightarrow \exists \text{prime } p.\ p | m_1 \wedge p | n_1$$

It follows

$$\exists \text{ prime } p.\ p | m_1 \wedge p | n_1$$

There is a prime p_1 such that $p_1 | m_1$ and $p_1 | n_1$; that is, there are pos. int. m_2 and n_2 such that

$$m_1 = p_1 m_2 \quad \text{and} \quad n_1 = p_1 \cdot n_2$$

In fact, we have

$$m_0 = p_0 \cdot p_1 \cdot m_2 \quad \text{and} \quad n_0 = p_0 \cdot p_1 \cdot n_2$$

Iterating this argument k times we have $m_0 = p_0 \cdot p_1 \cdots p_k m_k$

for primes $p_0, p_1 \cdots p_k$ and m_k a pos. int.

Therefore

$$m_0 \geq \underbrace{2 \cdot \ldots \cdot 2}_{k \text{ times}}$$

For $k = m_0$, we have

$$m_0 \geq 2^{m_0}$$

a contradiction ↯. $\boxtimes$

Every rational number can be expressed as a fraction in lowest terms