Negation

Negations are statements of the form

\[ \neg P \]

or, in other words,

\[ P \text{ is not the case} \]

or

\[ P \text{ is absurd} \]

or

\[ P \text{ leads to contradiction} \]

or, in symbols,

\[ \neg P \]
A first proof strategy for negated goals and assumptions:

If possible, reexpress the negation in an *equivalent* form and use instead this other statement.

**Logical equivalences**

\[
\begin{align*}
\neg(P \implies Q) & \iff P \land \neg Q \\
\neg(P \iff Q) & \iff P \iff \neg Q \\
\neg(\forall x. P(x)) & \iff \exists x. \neg P(x) \\
\neg(P \land Q) & \iff (\neg P) \lor (\neg Q) \\
\neg(\exists x. P(x)) & \iff \forall x. \neg P(x) \\
\neg(P \lor Q) & \iff (\neg P) \land (\neg Q) \\
\neg(\neg P) & \iff P \\
\neg P & \iff (P \implies \text{false})
\end{align*}
\]
Theorem 37  For all statements $P$ and $Q$, 

$$(P \implies Q) \implies (\neg Q \implies \neg P) \; .$$

**Proof:**  Let $P$ and $Q$ be statements.

Assume $P \implies Q$ \hspace{1cm} (1)

$RTP \quad \neg Q \implies \neg P$

Assume $\neg Q \iff (Q \implies false)$ \hspace{1cm} (2)

$RTP \quad \neg P \iff (P \implies false)$ \hspace{1cm} (3)

From (1) and (2) we have (3) as required \checkmark
NB:

Amongst the equivalences for negation we have postulated the somewhat controversial

\[
\neg \neg P \iff P
\]

which is classically accepted.
In this light, to prove \( P \), one may equivalently prove \((\neg P) \Rightarrow \text{false}\).

That is, assuming \(\neg P\) leads to contradiction. This technique is known as **proof by contradiction**.
Proof by contradiction

The strategy for proof by contradiction:

To prove a goal \( P \) by contradiction is to prove the equivalent statement \( \neg P \implies \text{false} \)

**Proof pattern:**

In order to prove \( P \)

1. **Write:** We use proof by contradiction. So, suppose \( P \) is false.

2. **Deduce a logical contradiction.**

3. **Write:** This is a contradiction. Therefore, \( P \) must be true.
Scratch work:

Before using the strategy

Assumptions Goal

::

After using the strategy

Assumptions Goal

contradiction

::

¬P
Theorem 39  For all statements $P$ and $Q$,

$$(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q) .$$

Proof: Let $P$ and $Q$ be statements.

Assume $\neg Q \Rightarrow \neg P$ (1)

$\neg Q \Rightarrow \neg P$

Assume $P$

$\neg Q \Rightarrow \neg P$

Assume $\neg Q$ (by contradiction) (2)

$\neg Q$

Assume $\neg Q$

$\neg Q$

From (1) and (2), we deduce $\neg P \iff (P \Rightarrow \text{false})$ (3)
From (3) and (4), _false_ follows as required.
\[ \text{NB:} \]
We have proved
\[(P \implies Q) \iff (\neg Q \implies \neg P)\]
which, in fact, we have already used as the technique of

Proof by the Contrapositive
Lemma 41  A positive real number $x$ is rational iff

$$\exists \text{ positive integers } m, n :$$
$$x = m/n \land \neg (\exists \text{ prime } p : p \mid m \land p \mid n)$$

(†)

**Proof:** Let $x$ be a positive real number.

(†) ⇒ $x$ is rational:

Assume (†). Then $x = m/n$ for some pos. int $m$ and $n$; therefore rational.

$x$ rational ⇒ (†)
\[ (+) \ \exists \text{ pos. int. } m, n. \]

\[ x = \frac{m}{n} \land \lnot (\exists \text{ prime } p. \ p|m \land p|n) \]

\[ \lnot (+) \iff \forall \text{ pos. int. } m, n. \]

\[ \lnot [x = \frac{m}{n} \land \lnot (\exists \text{ prime } p. \ p|m \land p|n)] \]

\[ \iff \forall \text{ pos. int. } m, n. \]

\[ \lnot (x = \frac{m}{n}) \lor (\exists \text{ prime } p. \ p|m \land p|n) \]

\[ \iff \forall \text{ pos. int. } m, n. \]

\[ (x = \frac{m}{n} \Rightarrow \exists \text{ prime } p. \ p|m \land p|n) \]
* x rational \implies (1)

Assume (1) \( x = \frac{m_0}{n_0} \) for some pos. int. \( m_0, n_0 \).

Assume (*) \( \forall \text{ pos. int. } m, n : \)

\[
x = \frac{m}{n} \implies \exists \text{ prime } p : p | m \land p | n.
\]

We need deduce a contradiction.

Instantiating (*), we have:

(2) \( x = \frac{m_0}{n_0} \implies \exists \text{ prime } p : p | m_0 \land p | n_0 \).

From (1) and (2), we have

(3) \( \exists \text{ prime } p. \ p | m_0 \land p | n_0 \).
There is a prime \( p \) such that \( p \mid m_0 \) and \( p \mid n_0 \).

That is, there are positive integers \( m_1 \) and \( n_1 \) such that

\[
\begin{align*}
  m_0 &= p_0 \cdot m_1 \\
  n_0 &= p_0 \cdot n_1
\end{align*}
\]

Also

\[ x = \frac{m_1}{n_1} \]

Instantiating (*)

\[
x = \frac{m_1}{n_1} \Rightarrow \exists \text{ prime } p. \ p \mid m_1 \land p \mid n_1
\]

It follows

\[
\exists \text{ prime } p. \ p \mid m_1 \land p \mid n_1
\]
There is a prime $p_1$ such that $p_1 | m_1$ and $p_1 | n_1$; that is, there are pos. int. $m_2$ and $n_2$ such that

$$m_1 = p_1 \cdot m_2 \quad \text{and} \quad n_1 = p_1 \cdot n_2$$

In fact, we have

$$m_0 = p_0 \cdot p_1 \cdot m_2 \quad \text{and} \quad n_0 = p_0 \cdot p_1 \cdot n_2$$

Iterating this argument $k$ times we have $m_0 = p_0 \cdot p_1 \cdots p_k \cdot m_k$ for primes $p_0, p_1, \ldots, p_k$ and $m_k$ a pos. int.
Therefore
\[ m_0 \geq 2 \cdots 2 \underbrace{2 \cdots 2}_k \text{ times} \]

For \( k = m_0 \), we have
\[ m_0 \geq 2^{m_0} \]
a contradiction.
Every rational number can be expressed as a fraction in lowest terms