EXISTENTIAL QUANTIFICATION

- How to PROVE them as goals.

&

- How to USE them as assumptions.

Existential quantification

Existential statements are of the form

> **there exists** an individual x in the universe of discourse for which the property $P(x)$ holds

or, in other words,

> **for some** individual x in the universe of discourse, the property $P(x)$ holds

or, in symbols,

$$\boxed{\exists x.\, P(x)}$$

equivalently

$\exists y.\, P(y)$

$\exists z.\, P(z)$

etc.

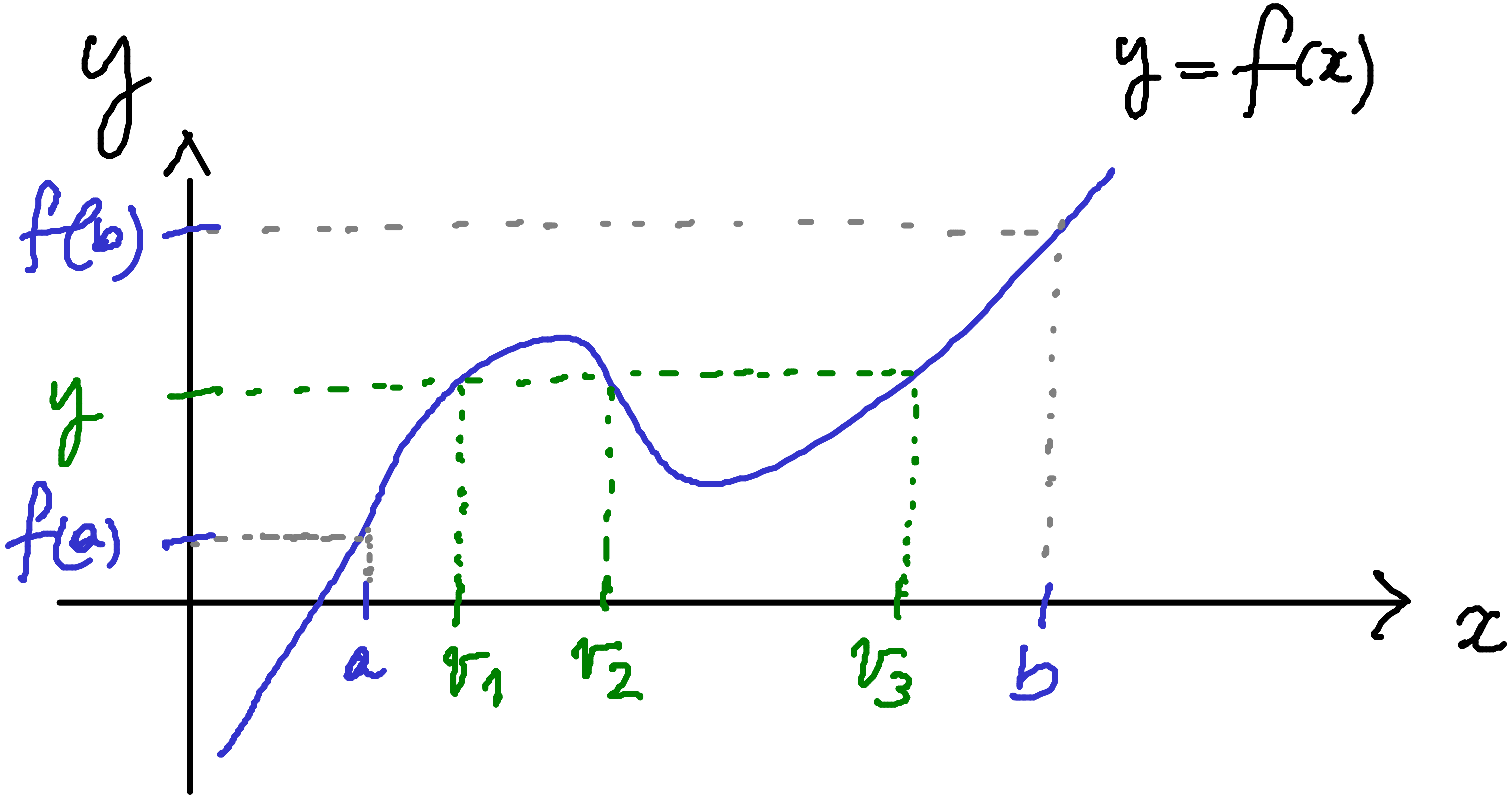
for all positive integers n and for all natural numbers $p_1, p_2, \dots, p_n$ if $p_1 + p_2 + \dots + p_n = n+1$ then there exists a positive integer i less than or equal n such that p_i is greater than 1.

Example: The Pigeonhole Principle.

Let n be a positive integer. If $n+1$ letters are put in n pigeonholes then there will be a pigeonhole with more than one letter.

Theorem 21 (Intermediate value theorem) *Let* f *be a real-valued continuous function on an interval* $[a, b]$*. For every* y *in between* $f(a)$ *and* $f(b)$*, there exists* v *in between* a *and* b *such that* $f(v) = y$*.*

Intuition:



The main proof strategy for existential statements:

To prove a goal of the form

$$\exists x.\, P(x)$$

find a ***witness*** for the existential statement; that is, a value of x, say w, for which you think $P(x)$ will be true, and show that indeed $P(w)$, i.e. the predicate $P(x)$ instantiated with the value w, holds.

Proof pattern:

In order to prove

$$\exists x.\, P(x)$$

1. Write: Let $w = \ldots$ (the witness you decided on).
2. Provide a proof of $P(w)$.

Scratch work:

Before using the strategy

Assumptions	Goal
	$\exists x.\, P(x)$
$\vdots$	

After using the strategy

Assumptions	Goals
	$P(w)$
$\vdots$	

$w = \ldots$ (the witness you decided on)

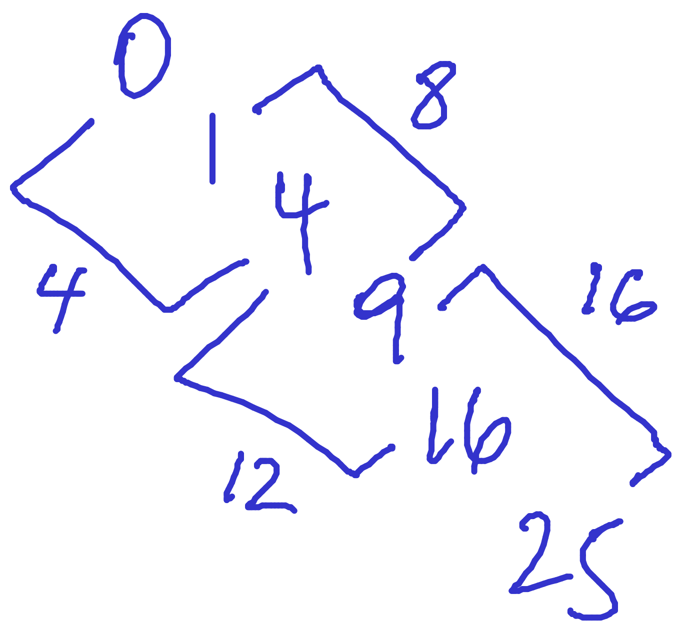
Proposition 22 *For every positive integer* k, *there exist natural numbers* i *and* j *such that* $4 \cdot k = i^2 - j^2$.

PROOF: Given a positive integer k we need find natural numbers i and j such that $4k = i^2 - j^2$.

k	$4k$	i	j
1	4	2	0
2	8	3	1
3	12	4	2
4	16	5	3

$j \xrightarrow{+1} i$, $i \xrightarrow{-1} j$

Given k
we guess
$i = k+1$
$j = k-1$



Proposition 22 *For every positive integer* k*, there exist natural numbers* i *and* j *such that* $4 \cdot k = i^2 - j^2$.

PROOF: Let k be a positive integer.

<u>RTP</u> $\exists$ nat. numbers i, j. $4k = i^2 - j^2$.

Let $i = k+1$ and $j = k-1$.

<u>RTP</u>: $4k = i^2 - j^2$

Indeed, $i^2 - j^2 = (k+1)^2 - (k-1)^2 = 4k$ $\boxtimes$

USE OF EXISTENTIAL ASSUMPTIONS

SCRATCH WORK

Before using the strategy

Assumptions

⋮

$\exists x.\ P(x)$

Goal

After using the strategy

Assumptions

⋮

$\exists x.\ P(x)$

$P(x_0)$ $\boxed{\text{for a new or fresh } x_0}$

Goal

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non sense

Goal

$\forall x. (\exists y. y=0) \Rightarrow x=0$

Assumptions

Let x be arbitrary

Goal

$(\exists y. y=0) \Rightarrow x=0$

Assumptions

Let x be arbitrary

$\exists y. y=0$

$x=0$

Goal

$x=0$

non sense

Goal

$\forall x. (\exists y. y=0) \Rightarrow x=0$

Assumptions

Let x be arbitrary

Goal

$(\exists y. y=0) \Rightarrow x=0$

Assumptions

Let x be arbitrary

$\exists y. y=0$

$y=0$

$z=0$

$\vdots$

Goal

$x=0$

The use of existential statements:

To use an assumption of the form $\exists x.\, P(x)$, introduce a new variable x_0 into the proof to stand for some individual for which the property $P(x)$ holds. This means that you can now assume $P(x_0)$ true.

Theorem 24 *For all integers* l, m, n, *if* $l \mid m$ *and* $m \mid n$ *then* $l \mid n$.

PROOF: Let ℓ, m, and n be integers.

Assume that $\ell | m$ and $m | n$.

That is:

① $\exists$ int. i s.t. $m = i \cdot \ell$

and ② $\exists$ int. j s.t. $n = j \cdot m$

Using ①, $m = i \cdot \ell$

Using ②, $n = jm$

Then, $n = j \cdot m = (j \cdot i) \cdot \ell$

Hence,

there exists an int. k, namely $k = j \cdot i$,

such that $n = k \cdot l$.

That is,

$$l \mid n$$

as required. $\boxtimes$