EXISTENTIAL QUANTIFICATION

• How to prove them as goals.
• How to use them as assumptions.
Existential quantification

Existential statements are of the form

**there exists** an individual \( x \) in the universe of discourse for which the property \( P(x) \) holds

or, in other words,

**for some** individual \( x \) in the universe of discourse, the property \( P(x) \) holds

or, in symbols,

\[
\exists x. P(x)
\]
for all positive integers n and for all natural numbers $p_1, p_2, \ldots, p_n$ if

\[ p_1 + p_2 + \ldots + p_n = n + 1 \]

then there exists a positive integer $i$ less than or equal to $n$ such that $p_i$ is greater than 1.

**Example:** The Pigeonhole Principle.

Let $n$ be a positive integer. If $n + 1$ letters are put in $n$ pigeonholes then there will be a pigeonhole with more than one letter.
Theorem 21 (Intermediate value theorem) Let $f$ be a real-valued continuous function on an interval $[a, b]$. For every $y$ in between $f(a)$ and $f(b)$, there exists $v$ in between $a$ and $b$ such that $f(v) = y$.

Intuition:
The main proof strategy for existential statements:

To prove a goal of the form

$$\exists x. P(x)$$

find a witness for the existential statement; that is, a value of $x$, say $w$, for which you think $P(x)$ will be true, and show that indeed $P(w)$, i.e. the predicate $P(x)$ instantiated with the value $w$, holds.
Proof pattern:
In order to prove

\[ \exists x. P(x) \]

1. **Write:** Let \( w = \ldots \) (the witness you decided on).
2. **Provide a proof of** \( P(w) \).
Scratch work:

Before using the strategy

Assumptions

Goal

$\exists x. P(x)$

After using the strategy

Assumptions

Goals

$P(w)$

$w = \ldots$ (the witness you decided on)
Proposition 22  For every positive integer $k$, there exist natural numbers $i$ and $j$ such that $4 \cdot k = i^2 - j^2$.

**Proof:** Given a positive integer $k$, we need to find natural numbers $i$ and $j$ such that $4k = i^2 - j^2$.

Given $k$, we guess:

$$i = k + 1$$

$$j = k - 1$$
Proposition 22  For every positive integer \( k \), there exist natural numbers \( i \) and \( j \) such that \( 4 \cdot k = i^2 - j^2 \).

**Proof:** Let \( k \) be a positive integer.

**RTP** \( \exists \) nat. numbers \( i, j \). \( 4k = i^2 - j^2 \).

Let \( i = k+1 \) and \( j = k-1 \).

**RTP:** \( 4k = i^2 - j^2 \)

Indeed, \( i^2 - j^2 = (k+1)^2 - (k-1)^2 = 4k \) \( \square \)
USE OF EXISTENTIAL ASSUMPTIONS

SCRATCH WORK

Before using the strategy:

Assumptions:

\[ \exists x. P(x) \]

After using the strategy:

Assumptions:

\[ \exists x. P(x) \]

\[ P(x_0) \] for a new or fresh \( x_0 \)

Goal ---

Goal ---
Assumptions
Let $x$ be arbitrary

\[
\exists y. y = 0
\]

$x = 0$

Goal

\[
\forall x. (\exists y. y = 0) \Rightarrow x = 0
\]
Assumptions

Let $x$ be arbitrary.

\[ \exists y. y = 0 \]

\[ y = 0 \]

\[ z = 0 \]

\[ \therefore x = 0 \]

Goal

\[ \forall x. (\exists y. y = 0) \Rightarrow x = 0 \]
The use of existential statements:

To use an assumption of the form $\exists x. P(x)$, introduce a new variable $x_0$ into the proof to stand for some individual for which the property $P(x)$ holds. This means that you can now assume $P(x_0)$ true.
Theorem 24  For all integers \( l, m, n \), if \( l \mid m \) and \( m \mid n \) then \( l \mid n \).

**Proof:** Let \( l, m, \) and \( n \) be integers.
Assume that \( l \mid m \) and \( m \mid n \).
That is:

1. \( \exists \text{ int. } i \text{ s.t. } m = i \cdot l \)
2. \( \exists \text{ int. } j \text{ s.t. } n = j \cdot m \)

Using (1), \( m = i \cdot l \)
Using (2), \( n = j \cdot m \)

Then, \( n = j \cdot m = (j \cdot i) \cdot l \)
Hence,

there exists an int. $k$, namely $k = \text{int. }$, such that $n = k \cdot e$.

That is,

$e/n$

as required.