IMPLICATIONS

- How to PROVE them as goals.

&

- How to USE them as assumptions.

Implication

Theorems can usually be written in the form

> **if** a collection of *assumptions* holds,
> **then** so does some *conclusion*

or, in other words,

> a collection of *assumptions* **implies** some *conclusion*

or, in symbols,

> a collection of *hypotheses* $\Longrightarrow$ some *conclusion*

NB Identifying precisely what the assumptions and conclusions are is the first goal in dealing with a theorem.

HOW TO PROVE IMPLICATION GOALS

The main proof strategy for implication:

To prove a goal of the form

$$P \implies Q$$

assume that P is true and prove Q.

NB *Assuming* is not *asserting*! Assuming a statement amounts to the same thing as adding it to your list of hypotheses.

Proof pattern:

In order to prove that

$$\mathrm{P} \implies \mathrm{Q}$$

1. Write: $\text{Assume } \mathrm{P}$.
2. Show that Q logically follows.

Scratch work:

Before using the strategy

Assumptions	Goal
	$P \implies Q$
$\vdots$	

After using the strategy

Assumptions	Goal
	Q
$\vdots$	
P	

Proposition 8 *If* m *and* n *are odd integers, then so is* $m \cdot n$.

PROOF:

Assume m and n are odd integers

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An alternative proof strategy for implication:

To prove an implication, prove instead the equivalent statement given by its contrapositive.

Definition:

the *contrapositive* of ‘P implies Q’ is ‘not Q implies not P’

WARNING

It is a frequent mistake for students to mix up and instead prove

The negation of P implies the negation of Q

So be very careful when using this technique!

Proof pattern:

In order to prove that

$$P \implies Q$$

1. Write: We prove the contrapositive; that is, ... and state the contrapositive.
2. Write: Assume 'the negation of Q'.
3. Show that 'the negation of P' logically follows.

Scratch work:

Before using the strategy

Assumptions	Goal
	$P \implies Q$
$\vdots$	

After using the strategy

Assumptions	Goal
	not P
$\vdots$	
not Q	

Definition 9 *A real number is:*

- rational *if it is of the form* m/n *for a pair of integers* m *and* n*; otherwise it is* irrational*.*
- positive *if it is greater than* 0*, and* negative *if it is smaller than* 0*.*
- nonnegative *if it is greater than or equal* 0*, and* nonpositive *if it is smaller than or equal* 0*.*
- natural *if it is a nonnegative integer.*

Proposition 10 *Let x be a positive real number. If x is irrational then so is $\sqrt{x}$.*

PROOF: ~~Let x be a positive real number.~~

~~Assume x is irrational, i.e, it cannot be expressed as a fraction.~~

~~RTP: $\sqrt{x}$ cannot be expressed as a fraction.~~

Proposition 10 *Let* x *be a positive real number. If* x *is irrational then so is* $\sqrt{x}$.

PROOF: Let x be a positive real number.
We equivalently prove, by the contrapositive, that
If $\sqrt{x}$ is rational then x is rational.
Assume $\sqrt{x}$ is rational; that is, of the form m/n for some integers m and n.
Then, $x = m^2/n^2$ is a fraction, and hence rational. $\boxtimes$

How to use implication assumptions

Logical Deduction
— Modus Ponens —

A main rule of *logical deduction* is that of *Modus Ponens*:

> From the statements P and $P \implies Q$,
> the statement Q follows.

or, in other words,

> If P and $P \implies Q$ hold then so does Q.

or, in symbols,

$$\frac{P \qquad P \implies Q}{Q}$$

The use of implications:

To use an assumption of the form $P \implies Q$,
aim at establishing P.
Once this is done, by Modus Ponens, one can
conclude Q and so further assume it.

Theorem 11 *Let* P_1, P_2, *and* P_3 *be statements. If* $P_1 \implies P_2$ *and* $P_2 \implies P_3$ *then* $P_1 \implies P_3$.

PROOF: Let P_1, P_2, P_3 be statements.

Assume: (1) $P_1 \Rightarrow P_2$ (2) $P_2 \Rightarrow P_3$

RTP: $P_1 \Rightarrow P_3$

Assume: (3) P_1

RTP: P_3

From (1) & (3), by MP, we deduce (4) P_2.

From (4) & (2), by MP, we deduce P_3.

$\boxtimes$

Theorem For $P, Q,$ and R statements,

$$(P \Rightarrow Q) \Rightarrow \left[\left(P \Rightarrow (Q \Rightarrow R) \right) \Rightarrow (P \Rightarrow R) \right]$$

PROOF: