Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

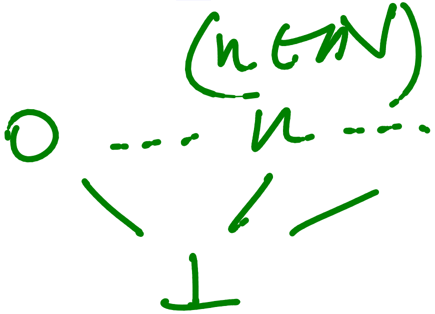
between domains.

Denotational semantics of PCF types

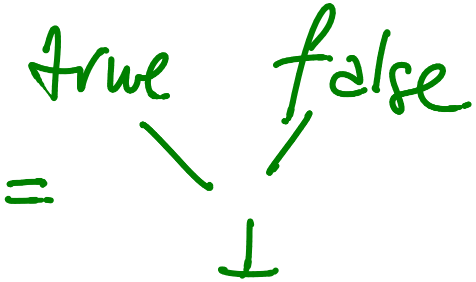
$[\![nat]\!] \overset{\text{def}}{=} \mathbb{N}_\perp$ (flat domain)

$[\![bool]\!] \overset{\text{def}}{=} \mathbb{B}_\perp$ (flat domain)

$[\![nat]\!] =$



$[\![bool]\!] =$



where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF types

$$[\![nat]\!] \overset{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \overset{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \overset{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF type environments

$$\llbracket \Gamma \rrbracket \overset{\text{def}}{=} \prod_{x \in dom(\Gamma)} \underbrace{\llbracket \Gamma(x) \rrbracket}_{\text{a domain}} \qquad (\Gamma\text{-environments})$$

A type environment Γ is a partial function from variables to types with finite domain of definition

$$[\![\Gamma]\!] = \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$

- elements

$$\underbrace{\rho}_{\Gamma\text{-environments}} = \{ \rho(x) \in [\![\Gamma(x)]\!] \}_{x \in \underline{dom}(\Gamma)}$$

- order

$$\rho \sqsubseteq \rho'$$

iff

$$\forall x \in \underline{dom}(\Gamma).\ \rho(x) \sqsubseteq \rho'(x)$$

in $[\![\Gamma(x)]\!]$

$[\![\Gamma]\!]$ is a domain

- Least element

$$\bot = \{ \bot_{[\![\Gamma(x)]\!]} \}_{x \in \underline{dom}(\Gamma)}$$

- Lubs

$$\left(\bigsqcup_n \rho n\right)(x) = \bigsqcup_n \rho_n(x) \text{ in } [\![\Gamma(x)]\!]$$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \overset{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad (\Gamma\text{-environments})$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \overset{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(Γ-environments)}$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

Example:

1. For the empty type environment $\emptyset$,

$$[\![\emptyset]\!] = \{\perp\}$$

where $\perp$ denotes the unique partial function with $dom(\perp) = \emptyset$.

2. $[\![\langle x \longmapsto \tau \rangle]\!] \; = \; \big(\{\, x \,\} \to [\![\tau]\!]\big)$

2. $[\![\langle x \mapsto \tau \rangle]\!] = \left(\{\, x \,\} \to [\![\tau]\!]\right) \cong [\![\tau]\!]$

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3.

$$
\begin{aligned}
&[\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
&\quad \cong (\{ x_1 \} \to [\![\tau_1]\!]) \times \ldots \times (\{ x_n \} \to [\![\tau_n]\!]) \\
&\quad \cong [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

Denotational Semantics of terms

Recall that for $\Gamma \vdash M : \tau$ we aim to compositionally define a continuous function $[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$.

We proceed by induction on the structure of terms, giving

$$[\![\Gamma \vdash M]\!](\rho) \in [\![\tau]\!] \qquad \text{for } \rho \in [\![\Gamma]\!]$$

Denotational semantics of PCF terms, I

$$\llbracket \Gamma \vdash \mathbf{0} \rrbracket(\rho) \stackrel{\text{def}}{=} 0 \in \llbracket nat \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{true} \rrbracket(\rho) \stackrel{\text{def}}{=} true \in \llbracket bool \rrbracket$$

$$\llbracket \Gamma \vdash \mathbf{false} \rrbracket(\rho) \stackrel{\text{def}}{=} false \in \llbracket bool \rrbracket$$

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad (x \in dom(\Gamma))$$

NB. $[\![\Gamma \vdash x]\!] : [\![\Gamma]\!] \to [\![\Gamma(x)]\!]$ is a projection function and hence continuous

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

<u>NB</u>: $[\![\Gamma \vdash \underline{succ}(M)]\!] = s \circ [\![\Gamma \vdash M]\!]$

where $s : \mathbb{N}_\bot \longrightarrow \mathbb{N}_\bot$

$\bot \longmapsto \bot$

$\mathbb{N} \ni n \longmapsto n+1$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

NB: $[\![\Gamma \vdash \underline{pred}(M)]\!] = p \circ [\![\Gamma \vdash M]\!]$

where $p : \mathbb{N}_\bot \longrightarrow \mathbb{N}_\bot$

$$\bot, 0 \longmapsto \bot$$

$$\mathbb{N} \ni n+1 \longmapsto n$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

Exercise: Define a continuous function $z : \mathbb{N}_\bot \to \mathbb{B}_\bot$

$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

such that $[\![\Gamma \vdash \underline{\mathrm{zero}}(M)]\!] = z \circ [\![\Gamma \vdash M]\!]$

NB:

$$[\![\Gamma \vdash op(M)]\!] = \underbrace{f}_{\text{continuous}} \circ [\![\Gamma \vdash M]\!]$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \textbf{if } M_1 \textbf{ then } M_2 \textbf{ else } M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \perp & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \perp \end{cases}$$

<u>NB</u>: $\mathit{cond} : \mathbb{B}_\perp \times D \times D \longrightarrow D$ is continuous.

$$\begin{aligned} \perp \quad d_1 \quad d_2 &\longmapsto \perp \\ \mathit{true} \quad d_1 \quad d_2 &\longmapsto d_1 \\ \mathit{false} \quad d_1 \quad d_2 &\longmapsto d_2 \end{aligned}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash M_1\, M_2]\!](\rho) \stackrel{\text{def}}{=} \left([\![\Gamma \vdash M_1]\!](\rho)\right)\left([\![\Gamma \vdash M_2]\!](\rho)\right)$$

<u>NB</u>: $(D \to E) \times D \longrightarrow E : (f, x) \mapsto f(x)$
is continuous

For $\Gamma \vdash \underline{fn}\ x:\tau.\ M : \tau \to \sigma$, we wish to define

$$[\![\Gamma \vdash \underline{fn}\ x:\tau.\ M]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\sigma]\!])$$

in terms of

$$[\![\Gamma[x \mapsto \tau] \vdash M : \sigma]\!] : [\![\Gamma[x \mapsto \tau]]\!] \to [\![\sigma]\!]$$

For $\rho \in [\![\Gamma]\!]$,

$$[\![\Gamma \vdash \underline{fn}\, x{:}\tau . M]\!](\rho) \in ([\![\tau]\!] \to [\![\sigma]\!])$$

$$\| \text{def}$$

$$\lambda d \in [\![\tau]\!] . \underbrace{[\![\Gamma[x \mapsto \tau] \vdash M]\!]\big(\overbrace{\rho[x \mapsto d]}^{\in [\![\Gamma[x \mapsto \tau]]\!]}\big)}_{\in [\![\sigma]\!]}$$

$$(\rho[x \mapsto d])(v) = \begin{cases} d & v = x \\ \rho(v) & \text{otw.} \end{cases}$$

Denotational semantics of PCF terms, IV

$$\begin{aligned}&[\![\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M]\!](\rho)\\ &\quad\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!] \,.\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d])\end{aligned} \qquad (x \notin dom(\Gamma))$$

NB: $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping x to $d \in [\![\tau]\!]$ and otherwise acting like ρ.

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

NB: $[\![\Gamma \vdash \underline{\mathit{fix}}(M)]\!] = \mathit{fix} \circ [\![\Gamma \vdash M]\!]$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{ \perp \}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

if $[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

then $\left[\!\!\left[\Gamma' \vdash \mathcal{C}[M]\right]\!\!\right] = \left[\!\!\left[\Gamma' \vdash \mathcal{C}[M]\right]\!\!\right] : [\![\Gamma']\!] \to [\![\tau']\!]$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$*,*

$$\text{if } M \Downarrow_\tau V \text{ then } [\![M]\!] = [\![V]\!] \in [\![\tau]\!] \ .$$

CASE

$$\frac{M \Downarrow \underline{\mathrm{succ}}(V)}{\underline{\underline{\mathrm{pred}}}(M) \Downarrow V}$$

By induction

$$[\![M]\!] = [\![\underline{\mathrm{succ}}(V)]\!] = [\![V]\!] + 1$$

<u>RTP</u> $[\![\mathrm{pred}(M)]\!] \overset{?}{=} [\![V]\!]$

$$\begin{array}{c} [\![\mathrm{pred}(M)]\!] \\ \shortparallel \\ p([\![M]\!]) = p([\![V]\!]+1) = [\![V]\!] \end{array}$$

✓

CASE

$$\frac{M\,(\underline{\underline{fix}}(M)) \Downarrow V}{\underline{\underline{fix}}(M) \Downarrow V}$$

By induction $[\![M\,(\underline{\underline{fix}}(M))]\!] = [\![V]\!]$

$$\begin{aligned}
&[\![M\,(\underline{\underline{fix}}(M))]\!] \\
&\quad \| \\
&[\![M]\!]\,([\![\underline{\underline{fix}}(M)]\!]) \\
&\quad \| \\
&[\![M]\!]\,(fix\,([\![M]\!])) \\
&\quad \| \\
&\underline{fix}\,[\![M]\!]
\end{aligned}$$

✓

CASE

$$\frac{M_1 \Downarrow \underline{fn}\ x:\tau.M \qquad\qquad M[M_2/x] \Downarrow V}{M_1\ M_2 \Downarrow V}$$

By induction

$$[\![M_1]\!] = [\![\underline{fn}\ x:\tau.M]\!]$$

$$= \lambda d \in [\![\tau]\!].\ [\![x \mapsto \tau \vdash M]\!]\,[x \mapsto d]$$

$$[\![M[M_2/x]\,]\!] = [\![V]\!]$$

RTP: $[\![M_1(M_2)]\!] \overset{?}{=} [\![V]\!]$

$$\|$$

$$[\![M_1]\!]\left([\![M_2]\!]\right)$$

$$\llbracket M_1\ M_2 \rrbracket = \llbracket x \mapsto \tau \vdash M \rrbracket \left[x \mapsto \llbracket M_2 \rrbracket \right]$$

$$\Vert\ ?$$

$$\llbracket v \rrbracket = \llbracket M \left[{}^{M_2}/_{x} \right] \rrbracket$$

Lemma

$$\llbracket x \mapsto \tau \vdash M \rrbracket \left[x \mapsto \llbracket M_2 \rrbracket \right]$$

$$= \llbracket M \left[{}^{M_2}/_{x} \right] \rrbracket$$

SUBSTITUTION
LEMMA

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*.*

Then,

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!](\rho)\big]\big)
\end{aligned}
$$

for all $\rho \in [\![\Gamma]\!]$*.*

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

CASE

$$\Gamma[x \mapsto \tau] \vdash \underbrace{\underline{fn}\, y : \sigma . N}_{= M'} : \underbrace{\sigma \to \sigma'}_{= \tau'}$$

RTP: for $\rho \in [\![\Gamma]\!]$

$$[\![\Gamma \vdash (\underline{fn}\, y : \sigma . N)[{}^{M}/_{x}]\,]\!](\rho)$$

$$\overset{?}{=}$$

$$[\![\Gamma[x \mapsto \tau] \vdash \underline{fn}\, y : \sigma . N\,]\!]\big(\rho[x \mapsto [\![\Gamma \vdash M]\!](\rho)]\big)$$

$$[\![\Gamma \vdash (\underline{\mathrm{fn}}\, y : \sigma . N)[{}^{M}/_{x}]]\!](\rho)$$

$$= [\![\Gamma \vdash \underline{\mathrm{fn}}\, y : \sigma . N[{}^{M}/_{x}]]\!](\rho)$$

$$= \lambda d \in [\![\sigma]\!] . [\![\Gamma[y \mapsto \sigma] \vdash N[{}^{M}/_{x}]]\!](\rho[y \mapsto d])$$

by ind.

$$= \lambda d \in [\![\sigma]\!] .$$

$$[\![\Gamma[y \mapsto \sigma][x \mapsto \tau] \vdash N]\!]$$

$$\left(\rho[y \mapsto d][x \mapsto [\![\Gamma[y \mapsto \sigma] \vdash M]\!](\rho[y \mapsto d])]\right)$$

$$[\![\Gamma[x \mapsto \tau] \vdash \underline{\mathrm{fn}}\, y : \sigma . N]\!] (\rho[x \mapsto [\![\Gamma \vdash M]\!](\rho)])$$

$$= \lambda d \in [\![\sigma]\!] .$$

$$[\![\Gamma[x \mapsto \tau][y \mapsto \sigma] \vdash N]\!]$$

$$(\rho[x \mapsto [\![\Gamma \vdash M]\!](\rho)][y \mapsto d])$$

for $d \in [\![\sigma]\!]$,

$$\begin{aligned}
&[\![\Gamma[y\mapsto\sigma][x\mapsto\tau]\vdash N]\!]\\
&\quad\left(\rho[y\mapsto d][x\mapsto [\![\Gamma[y\mapsto\sigma]\vdash M]\!](\rho[y\mapsto d])]\right)\\
\overset{?}{=}\;&\\
&[\![\Gamma[x\mapsto\tau][y\mapsto\sigma]\vdash N]\!]\\
&\quad\left(\rho[x\mapsto [\![\Gamma\vdash M]\!](\rho)][y\mapsto d]\right)
\end{aligned}$$

Weakening Lemma

Weakening Property

Proposition Suppose $\Gamma \vdash M : \tau$.

Then, for $y \notin \underline{\text{dom}}(\Gamma)$,

$$[\![\Gamma[y \mapsto \sigma] \vdash M]\!](\rho[y \mapsto d])$$

$$= [\![\Gamma \vdash M]\!](\rho)$$

for all $\rho \in [\![\Gamma]\!]$ and $d \in [\![\sigma]\!]$.

NB: One proves

- a weakening lemma

to prove

- a substitution lemma

to prove

- denotational soundness.