Topic 4

Scott Induction

Consider a continuous function $f: D \to D$ for D a domain.

For a subset $S \subseteq D$, representing a property of interest, we wish to show that the element

$$\underline{\mathrm{fix}}(f) \in D$$

recursively defined as the least such that

$$f(\underline{\mathrm{fix}}\, f) = \underline{\mathrm{fix}}(f)$$

satisfies the property; i.e. $\underline{\mathrm{fix}}(f) \in S$.

Suppose that

(A1) $\perp \in S$

and

(I) f has the property

$$\forall d \in D.\ d \in S \Rightarrow f(d) \in S$$

In other words, THE PROPERTY S IS INVARIANT UNDER f.

Then, by induction,

$$\forall n \in \mathbb{N},\ f^n(\perp) \in S.$$

Suppose further that

(A2) for every chain

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad (n \in \mathbb{N})$$

in S

we also have

$$\left(\bigsqcup_n d_n \right) \in S$$

Then,

$$\underline{fix}(f) = \bigsqcup_n f^n(\perp) \in S.$$

For $S \subseteq D$ satisfying (A1) and (A2) we have

$$\frac{\forall d \in D.\ d \in S \Rightarrow f(d) \in S}{\mathit{fix}(f) \in S} \quad (S \text{ admissible})$$

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n \geq 0} d_n\right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

The downset of d

of D is chain-closed.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

 of D is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

 and

$$\{(x, y) \in D \times D \mid x = y\}$$

 of $D \times D$ are chain-closed.

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

$f(d) \sqsubseteq d$

$x \sqsubseteq d \Rightarrow f(x) \sqsubseteq f(d)$

f monotone

$$\underbrace{\qquad\qquad\qquad\qquad}_{f(x) \sqsubseteq d}$$

$$\frac{\forall x \in D.\ x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}{\mathit{fix}(f) \sqsubseteq d} \quad (\downarrow(d) \text{ admissible})$$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in \downarrow(d) &\implies x \sqsubseteq d \\ &\implies f(x) \sqsubseteq f(d) \\ &\implies f(x) \sqsubseteq d \\ &\implies f(x) \in \downarrow(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in \downarrow(d) \ .$$

The property of an admissible subset S of a domain that

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad (n \in \mathbb{N}) \quad \text{in } S$$

implies

$$\left(\bigsqcup_n d_n \right) \quad \text{in } S$$

is called CHAIN-CLOSURE.

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

If f ($*$) $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \; (n \in \mathbb{N})$ in $f^{-1}(S)$

Then $\bigsqcup_n d_n$ is in $f^{-1}(S)$

($*$) $f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots \; (n \in \mathbb{N})$ in S

$$\left.\begin{array}{l} \Rightarrow \quad \bigsqcup_n f(d_n) \in S \\ \qquad\quad \parallel \\ \qquad f\left(\bigsqcup_n d_n\right) \end{array}\right\} \Rightarrow \left(\bigsqcup_n d_n\right) \in f^{-1}(S)$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

ADMISSIBLE ?

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \, .$$

$f \circ g \sqsubseteq g \circ f$

$f(\bot) \sqsubseteq g(\bot)$

$\mathit{fix}(g) \in \{ d \in D \mid f(d) \sqsubseteq g(d) \}$

$$\dfrac{\dfrac{f(\mathit{fix}\, g) \sqsubseteq g(\mathit{fix}\, g) \qquad \dfrac{}{g(\mathit{fix}(g)) \sqsubseteq \mathit{fix}(g)} \checkmark}{f(\mathit{fix}\, g) \sqsubseteq \mathit{fix}(g)}}{\mathit{fix}(f) \sqsubseteq \mathit{fix}(g)}$$

$S = \{ d \in D \mid f(d) \sqsubseteq g(d) \}$ admissible

$\perp \in S \iff f(\perp) \sqsubseteq g(\perp) \rightsquigarrow$ assumption

Define $\langle f, g \rangle : D \to D \times D$

$\qquad = \lambda d.\, (f(d), g(d))$

Observe

$$S = \langle f, g \rangle^{-1} (\sqsubseteq)$$

$$f(d) \sqsubseteq g(d)$$

$$g(f\,d) \sqsubseteq g(g\,d)$$

$$f \circ g \sqsubseteq g \circ f$$

$$\frac{f(g\,d) \sqsubseteq g(f\,d)}{f(g\,d) \sqsubseteq g(g\,d)}$$

$$\frac{\forall d \in D.\ f(d) \sqsubseteq g(d) \Longrightarrow f(g\,d) \sqsubseteq g(g\,d)}{f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g))}$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

If S and T are chain-closed then so is $S \cup T$.
Consider $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots (n \in \mathbb{N})$ in $S \cup T$.

(1) If $\{d_n \mid n \in \mathbb{N}\} \cap S$ infinite then
$\bigsqcup_n d_n = \left(\bigsqcup \{d_n \mid n \in \mathbb{N}\} \cap S\right) \in S \subseteq S \cup T$.

(2) If $\{d_n \mid n \in \mathbb{N}\} \cap T$ infinite Then
$\bigsqcup_n d_n = \left(\bigsqcup \{d_n \mid n \in \mathbb{N}\} \cap T\right) \in T \subseteq S \cup T$.

(3) If $\{d_n \mid n \in \mathbb{N}\} \cap S$ and $\{d_n \mid n \in \mathbb{N}\} \cap T$ finite Then $\bigsqcup_n d_n = d_N$ for $N \in \mathbb{N}$ and therefore it is either in S or T.

$\boxtimes$

Admissible subsets need not be closed under infinite unions.

$$\overline{\omega} = (0 \leq 1 \leq \cdots \leq n \leq \cdots \leq \infty)$$
$(n \in \mathbb{N})$

We have that, for all $i \in \mathbb{N}$,

$${\downarrow}(i) \text{ is admissible}$$

However, $\bigcup_{i \in \mathbb{N}} {\downarrow}(i)$ is not admissible.

- Convention: For $x, y \in \mathbb{Z}$, we write $[X \mapsto x, Y \mapsto y]$ or simply (x, y) for the state function from $\mathbb{L} = \{X, Y\}$ to $\mathbb{Z}$ mapping X to x and Y to y.

Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\mathbf{while}\ X > 0\ \mathbf{do}\ (Y := X * Y; X := X - 1)\ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$
$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$ is given by

$$f(w) = \lambda(x, y) \in \mathit{State}. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \ .$$

Admissibility of S

- $\perp \in S$ trivially
- Consider $w_0 \sqsubseteq w_1 \sqsubseteq \cdots \sqsubseteq w_n \sqsubseteq \cdots (n \in \omega)$ in S

Say $w = \bigsqcup_n w_n$

Suppose $w[X \mapsto x, Y \mapsto y]\downarrow$

Then there exists $n \in \omega$ such that $w_n[X \mapsto x, Y \mapsto y]\downarrow$ and equals $[X \mapsto 0, Y \mapsto x! y]$.

Moreover as $w[X \mapsto x, Y \mapsto y] = w_n[X \mapsto x, Y \mapsto y]$

we are done.

$\boxtimes$

Assume $w \in S$ $(*)$

<u>RTP</u>: $f(w) \in S$

$= \lambda(x,y).\,\mathrm{If}\left(x \le 0, (x,y), w(x-1, x\cdot y)\right)$

Let $x, y \geq 0$.

$f(w)(x,y)\downarrow \iff \mathrm{If}\left(x \le 0, (x,y)\right), w(x-1, x\cdot y))\downarrow$

<u>CASE $x > 0$</u>

$f(w)(x,y) = w(x-1, x\cdot y)\downarrow$

$\Rightarrow$ by $(*)$ $w(x-1, x\cdot y) = (0, (x-1)!\cdot x\cdot y)$

$= (0, x!\cdot y)$

<u>CASE $x = 0$</u>

$f(w)(0,y) = (0,y) = (0, 0!\cdot y)$

$\boxtimes$