# Topic 4

## Scott Induction

Consider à continuous function f: D->D for D2 domain. For a subset SSD, representing a property of interest, we wish to show That the élement FIX (F) ED recursively defined as The least such That  $f(f\alpha f) = f\alpha(f)$ satisfies the property; i.e. fract, ES.

Suppose That (A1) LES (I) f has the property (J) f has the property  $\forall d \in D. d \in S \Longrightarrow f(d) \in S$ Ind In other words, THE PROPERTY S TS INVARIANT UNDER f. men, by induction,  $\forall n \in \mathcal{N}, f^n(\mathcal{I}) \in S$ .

Suppose further That (A2) for every choon do 5 dr 5 --- 5 dn 5 --- (162) in S we dør hære (LIndn)ES Then,  $f_{\underline{n}}(f) = \bigcup_{n} f^{n}(I) \in S.$ 

For SSD satisfying (A1) and (A2) we have

Hdel). des => f(d) ∈ S fix (f) ∈ S

#### **Scott's Fixed Point Induction Principle**

Let  $f: D \to D$  be a continuous function on a domain D. For any <u>admissible</u> subset  $S \subseteq D$ , to prove that the least fixed point of f is in S, *i.e.* that

 $fix(f) \in S$ ,

it suffices to prove

 $\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$ 

Let D be a cpo. A subset  $S \subseteq D$  is called chain-closed iff for all chains  $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$  in D

$$(\forall n \ge 0 \, . \, d_n \in S) \implies \left(\bigsqcup_{n\ge 0} d_n\right) \in S$$

If D is a domain,  $S \subseteq D$  is called admissible iff it is a chain-closed subset of D and  $\bot \in S$ .

Let D, E be cpos.

#### **Basic relations:**

of D is chain-closed.

Let D, E be cpos.

#### **Basic relations:**

• For every  $d \in D$ , the subset

$$\downarrow(d) \stackrel{\mathrm{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

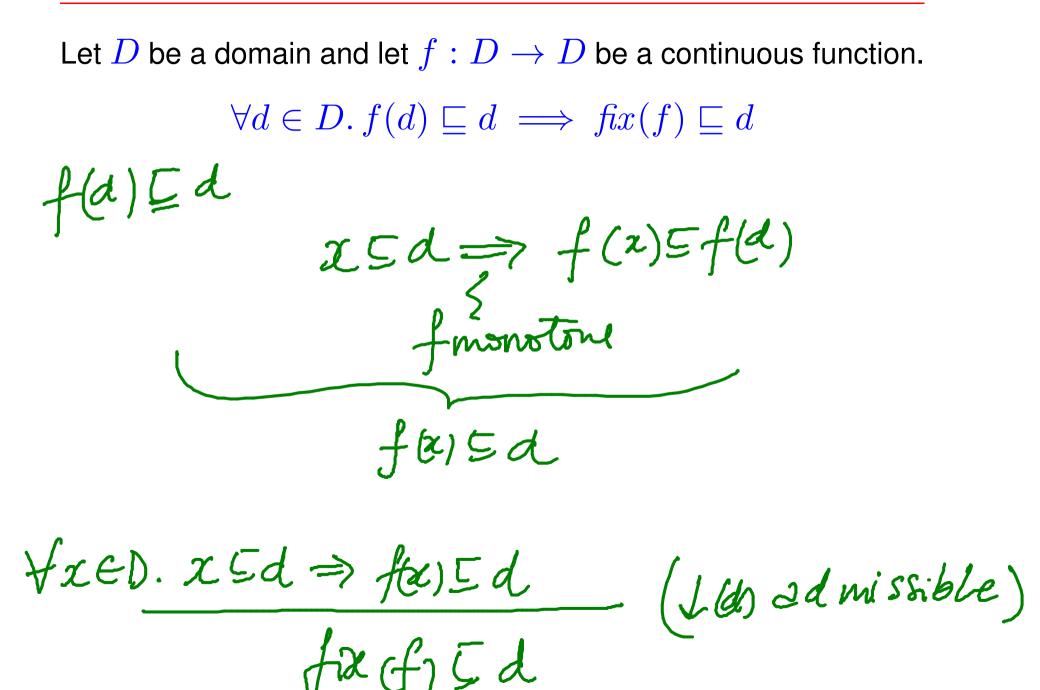
of D is chain-closed.

• The subsets

and  $\begin{cases} (x,y) \in D \times D \mid x \sqsubseteq y \\ \\ \{(x,y) \in D \times D \mid x = y \} \end{cases}$ 

of  $D \times D$  are chain-closed.

#### Example (I): Least pre-fixed point property



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Let D be a domain and let  $f: D \to D$  be a continuous function.  $\forall d \in D. f(d) \sqsubseteq d \implies fix(f) \sqsubseteq d$ 

Proof by Scott induction.

Let  $d \in D$  be a pre-fixed point of f. Then,

$$\begin{array}{rcl} x \in \downarrow(d) & \Longrightarrow & x \sqsubseteq d \\ & \Longrightarrow & f(x) \sqsubseteq f(d) \\ & \Longrightarrow & f(x) \sqsubseteq d \\ & \implies & f(x) \in \downarrow(d) \end{array}$$

Hence,

 $fix(f) \in {\downarrow}(d)$  .

$$do \equiv d_1 \equiv \dots \subseteq d_n \equiv \dots (n \in \mathbb{A})$$
 in S  
implies  
 $(\bigsqcup_n d_n)$  on S

is collect CHAIN-CLOSURE.

#### **Building chain-closed subsets (II)**

**Inverse image:** 

Let  $f: D \to E$  be a continuous function.

If S is a chain-closed subset of E then the inverse image

 $f^{-1}S = \{x \in D \mid f(x) \in S\}$ 

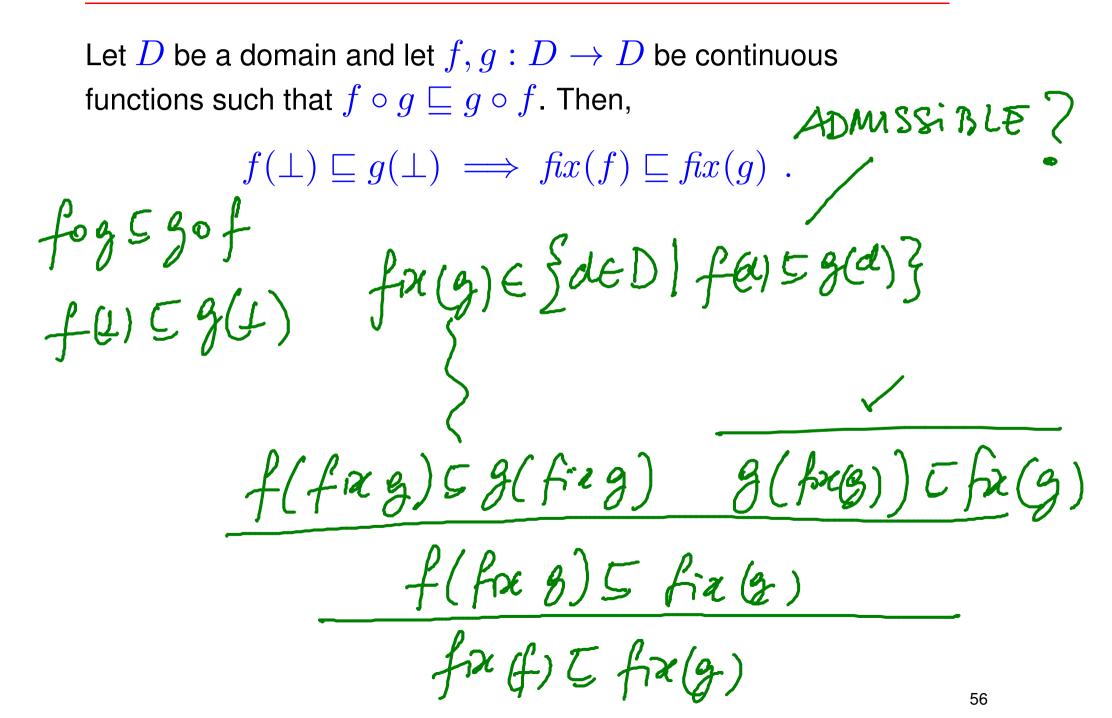
is an chain-closed subset of D.

Jf (\*)do5d15-- ⊑d1=---(ntal) inf<sup>-1</sup>(S)

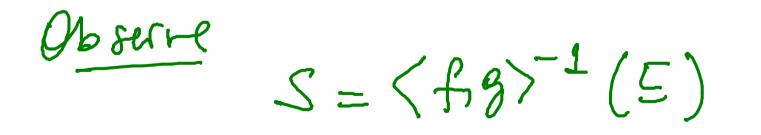
Then I don is in fl(S)

(\*) f(do) 5 f(d) 5 --- 5 f(du) 5 -- (n car) in S 

### Example (II)



S= {del | fle) 5g(d) } 2dmissible  $L \in S \Leftrightarrow f(t) \equiv g(t) \sim cossumption$ Define  $\langle f,g \rangle : D \rightarrow D \times D$   $\lim_{n \to \infty} d \cdot (f(d), g(d))$ 



f(a) 5 g(d)  $g(fd) \subseteq g(gd)$ fog 5 gof f(gd) 5 g(fd)  $f(gd) \equiv g(gd)$  $\forall d \in \mathcal{D}. f(d) \leq g(d) \Rightarrow f(gd) \leq g(gd)$  $f(f^{x}(g)) \subseteq g(f^{x}(g))$ 

Let D be a domain and let  $f, g : D \to D$  be continuous functions such that  $f \circ g \sqsubseteq g \circ f$ . Then,

$$f(\perp) \sqsubseteq g(\perp) \implies fix(f) \sqsubseteq fix(g)$$
.

Proof by Scott induction.

Consider the admissible property  $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$  of D.

Since

 $f(x)\sqsubseteq g(x) \Rightarrow g(f(x))\sqsubseteq g(g(x)) \Rightarrow f(g(x))\sqsubseteq g(g(x))$ 

we have that

$$f(fix(g)) \sqsubseteq g(fix(g))$$
.

#### Logical operations:

- If  $S, T \subseteq D$  are chain-closed subsets of D then  $S \cup T$  and  $S \cap T$  are chain-closed subsets of D.
- If  $\{S_i\}_{i \in I}$  is a family of chain-closed subsets of D indexed by a set I, then  $\bigcap_{i \in I} S_i$  is a chain-closed subset of D.
- If a property P(x, y) determines a chain-closed subset of  $D \times E$ , then the property  $\forall x \in D$ . P(x, y) determines a chain-closed subset of E.

If S and T are chosen - dored then si is SUT Consider do 5 dy 5 -- 5 dy 5 --- (n EN) in SUT. (1) If Edulnewigns infinite then  $i_n dn = (i_s d_n | n \in W_s^2 \cap S) \in S \subseteq S \cup T.$ (2) If § du/new 3 nT infinite Then Undr= (WEdn new 3 NT) ETE SUT. (3) If Edulne Nynsond Edulnen YNT finite then  $\Box_n dn = dw$  for  $w \in W$  and There fore it is either in S or T.



Admissible subsets need not be closed under infinite unions.  $\overline{w} = (0515 - ... 5n5 - ... (new))$ 50) We have that, for all icas, V(i) is admissible Honever, U I(i) 13 not admissible.

Let  $\mathcal{F}: State 
ightarrow State$  be the denotation of

while X > 0 do (Y := X \* Y; X := X - 1).

For all  $x, y \ge 0$ ,  $\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$  $\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$  Recall that

$$\mathcal{F} = fix(f)$$
  
where  $f: (State \rightarrow State) \rightarrow (State \rightarrow State)$  is given by  
$$f(w) = \lambda(x, y) \in State. \begin{cases} (x, y) & \text{if } x \leq 0\\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

#### Proof by Scott induction.

We consider the admissible subset of  $(State \rightarrow State)$  given by

$$S = \begin{cases} w & \forall x, y \ge 0. \\ w[X \mapsto x, Y \mapsto y] \downarrow \\ \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y] \end{cases}$$

and show that

 $w \in S \implies f(w) \in S$ .

Admissibility of S · LES trivielly • Consider  $w_0 \leq w_1 \leq \dots \leq w_n \leq \dots (n \in \omega)$  in S soy w= Unwn Suppose WEXMX, YMJJV. Then There exists near such That Wn [XH2, YHY] I and equals [XH0, YH2!y]. More over as w[XH32,YH3y]=wn[XH32,YHY] we are done.



Assume wes (\*) RTP: ftw)ES " $\lambda(x,y) \cdot f(x \leq 0, (x,y), w(x-1, x,y))$  $f(w)(x_iy) \downarrow \rightleftharpoons f(x \le 0, (x_iy), w(x-1, x \cdot y)) \downarrow$ CA8E 2>0  $f(w)(x,y) = w(x-1, x,y) \downarrow$  $w(x-1,x,y) = (0,(z-1)!\cdot x\cdot y)$ =) by (\*)  $=(0, x! \cdot y)$ CASE  $\chi = 0$ f(w)(o,y) = (o,y) = (o,o:-y)X