Topic 3

Constructions on Domains

Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

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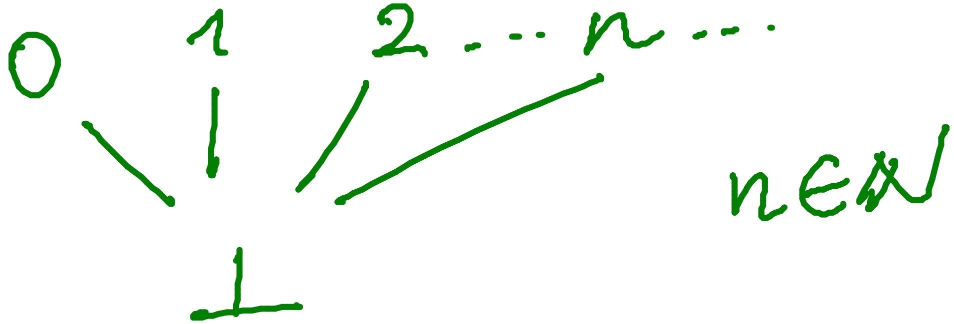
Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

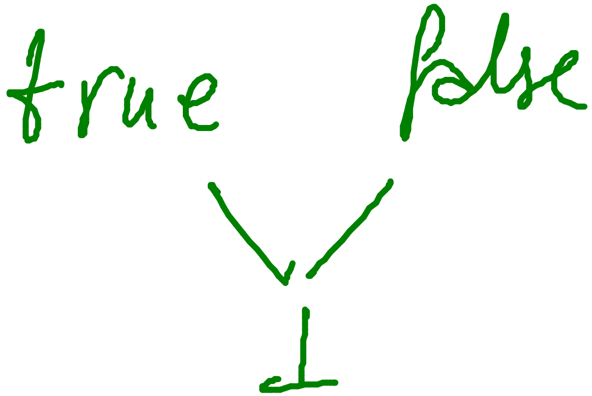
makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Examples:

$\mathbb{N}_\perp$



$\mathbb{B}_\perp$



PRODUCT TYPES

$$\frac{T_1 \text{ type} \qquad T_2 \text{ type}}{T_1 * T_2 \text{ type}}$$

The product of two domains $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ is the domain defined as:

- <u>underlying set</u>:

$$D \times E = \{ (d,e) \mid d \in D, e \in E \}$$

- <u>order</u>

$$(d,e) \sqsubseteq (d',e') \overset{\text{def}}{\iff} d \sqsubseteq_D d' \wedge e \sqsubseteq_E e'$$

- least element

$$(\bot_D, \bot_E)$$

- lubs

$$\bigsqcup_n (d_n, e_n) = \left(\bigsqcup_n d_n, \bigsqcup_n e_n \right)$$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \&\ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \ \&\ d_2 \sqsubseteq_2 d_2' \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

FUNCTIONS WITH MANY ARGUMENTS

A continuous function of two arguments, say in a domain D and a domain E, and values in a domain F is simply a continuous function from the domain $(D \times E)$ to the domain F.

Continuous functions of two arguments

Proposition. *Let* D, E, F *be cpo's. A function* $f : (D \times E) \to F$ *is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d, e) \sqsubseteq f(d', e)$$
$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d, e) \sqsubseteq f(d, e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m \, , \, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$
$$f(d \, , \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$f : (D \times E) \to F$ is monotone

Iff

$$d \sqsubseteq d' \Rightarrow f(d,e) \sqsubseteq f(d',e) \qquad (1)$$

$\wedge$

$$e \sqsubseteq e' \Rightarrow f(d,e) \sqsubseteq f(d,e') \qquad (2)$$

$(\Longrightarrow)$ $d \sqsubseteq d' \Rightarrow (d,e) \sqsubseteq (d',e) \Rightarrow f(d,e) \sqsubseteq f(d',e)$.

$(\Longleftarrow)$ $(d,e) \sqsubseteq (d',e') \overset{?}{\Longrightarrow} f(d,e) \sqsubseteq f(d',e')$

$\Downarrow$

by (2)

$d \sqsubseteq d'$ $\qquad$ $e \sqsubseteq e' \Rightarrow f(d',e) \sqsubseteq f(d',e')$

by (1) $\Downarrow$

$f(d,e) \sqsubseteq f(d',e)$

$\boxtimes$

iff $f : (D \times E) \to F$ preserves lubs

$$f\left(\bigsqcup_n d_n, e\right) = \bigsqcup_n f(d_n, e)$$

$$\wedge\ f\left(d, \bigsqcup_n e_n\right) = \bigsqcup_n f(d, e_n)$$

($\Rightarrow$) In $D \times E$ we have

$$\left(\bigsqcup_n d_n, e\right) = \left(\bigsqcup_n d_n, \bigsqcup_n e\right) = \bigsqcup_n (d_n, e)$$

So, as f preserves lubs,

$$f\left(\bigsqcup_n d_n, e\right) = f\left(\bigsqcup_n (d_n, e)\right) = \bigsqcup_n f(d_n, e)$$

$$f : (D \times E) \to F \text{ preserves lubs}$$

iff

$$f\left(\bigsqcup_n d_n, e\right) = \bigsqcup_n f(d_n, e) \qquad (1)$$

$$\wedge \quad f\left(d, \bigsqcup_n e_n\right) = \bigsqcup_n f(d, e_n) \qquad (2)$$

$(\Leftarrow)$

$$f\left(\bigsqcup_n (d_n, e_n)\right) \overset{?}{=} \bigsqcup_n f(d_n, e_n)$$

$$\begin{aligned} &\quad\ \ \| \\ f\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) &\overset{\text{by}(1)}{=} \bigsqcup_n f\left(d_n, \bigsqcup_m e_m\right) \\ &\overset{\text{by}(2)}{=} \bigsqcup_n \bigsqcup_m f(d_n, e_m) \\ &= \bigsqcup_k f(d_k, e_k). \quad \boxtimes \end{aligned}$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ continuous})$$

FUNCTION TYPES

$$\frac{T_1 \text{ Type} \qquad T_2 \text{ type}}{T_1 \to T_2 \text{ type}}$$

The domain of functions from a domain $(D, \sqsubseteq_D)$ to a domain $(E, \sqsubseteq_E)$ is defined as consisting of

- the underlying set

$$(D \to E) = \{ f \mid f \text{ is a continuous function from } D \text{ to } E \}$$

- and the partial order

$$f \sqsubseteq g \iff_{\text{def}} \forall d \in D.\, f(d) \sqsubseteq_E g(d).$$

- <u>least element</u>

$$\perp_{D\to E} : D \to E$$

$$\perp_{D\to E}(d) = \perp_E \qquad d \in D$$

i.e. $\perp_{D\to E} = \lambda d \in D.\ \perp_E$

- lubs of chains

Given a chain

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \ (n \in \omega) \quad \text{in } (D \to E)$$

We have, for every $d \in D$, the following chain

$$f_0(d) \sqsubseteq f_1(d) \sqsubseteq \cdots \sqsubseteq f_n(d) \cdots \ (n \in \omega) \quad \text{in } E$$

and we can define a function from D to E by setting

$$f(d) = \bigsqcup_n f_n(d) \ ; \text{ i.e. } f = \lambda d \in D . \bigsqcup_n f_n(d) .$$

- monotonicity

Assume $d \sqsubseteq_D d'$ in D

$$\frac{\forall n \quad \dfrac{d \sqsubseteq_D d'}{f_n(d) \sqsubseteq f_n(d')}\ f_n \text{ monotone}}{\bigsqcup_n f_n(d) = f(d) \sqsubseteq_E f(d') = \bigsqcup_n f_n(d')}$$

- preservation of lubs

for a chain $d_0 \sqsubseteq_D d_1 \sqsubseteq_D \cdots \sqsubseteq_D d_n \sqsubseteq \cdots$ $(n \in \omega)$ in D,

$$f\left(\bigsqcup_n d_n\right) \stackrel{?}{=} \bigsqcup_n f(d_n)$$

$$f\left(\bigsqcup_n d_n\right) = \bigsqcup_m f_m\left(\bigsqcup_n d_n\right) = \bigsqcup_m \bigsqcup_n f_m(d_n)$$

$$\bigsqcup_n f(d_n) = \bigsqcup_n \bigsqcup_m f_m(d_n) = \bigsqcup_m \bigsqcup_n f_m(d_n)$$

f is a lub of the chain $f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$ $(n \in \mathbb{N})$
in $(D \to E)$

- it is an upper bound

$\forall d \in D.$ $\quad f_n(d) \sqsubseteq \bigsqcup_n f_n(d) = f(d)$

Hence $\quad f_n \sqsubseteq f$ in $(D \to E)$

- least upper bound
a continuous function $g : D \to E$ s.t. $f_n \sqsubseteq g$ $\forall n$

Then $\forall d \in D$

$$f_n(d) \sqsubseteq g(d)$$

Therefore

$$f(d) = \bigsqcup_n f_n(d) \sqsubseteq g(d)$$

So

$$f \sqsubseteq g \text{ in } (D \to E) \qquad \boxtimes$$

$$\bigsqcup_n f_n = \lambda d \in D. \bigsqcup_n f_n(d).$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n \geq 0} f_n \;=\; \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d) \;\;.$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D \to E}(d) = \perp_E$, all $d \in D$.

Proposition Let $f : (D \times E) \to F$ be a continuous function. Then, the <u>currying</u>

$$\hat{f} : D \to (E \to F),$$

$$\hat{f}(d) \stackrel{\text{def}}{=} \lambda e \in E.\, f(d, e) \qquad \forall\, d \in D$$

is a continuous function.

- <u>Monotonicity</u>

Assume $d \sqsubseteq d'$ in D

$$\dfrac{\dfrac{\dfrac{\dfrac{}{d \sqsubseteq d'} \qquad \dfrac{}{e \sqsubseteq e}}{(d,e) \sqsubseteq (d',e)}}{\forall e \in E \quad f(d,e) = \hat{f}(d)(e) \sqsubseteq \hat{f}(d')(e) = f(d',e)}\; f_{mon}}{\hat{f}(d) \sqsubseteq \hat{f}(d')}$$

- preservation of lubs

for a chain in D, $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

$$\hat{f}\left(\bigsqcup_n d_n\right) \stackrel{?}{=} \bigsqcup_n \hat{f}(d_n) \quad \text{in } (E \to F)$$

iff $\forall e \in E$

$$\hat{f}\left(\bigsqcup_n d_n\right)(e) \stackrel{?}{=} \left(\bigsqcup_n \hat{f}(d_n)\right)(e)$$

$$\hat{f}\left(\bigsqcup_n d_n\right)(e) = f\left(\bigsqcup_n d_n, e\right)$$

$$\left(\bigsqcup_n \hat{f}(d_n)\right)(e) = \bigsqcup_n \left(\hat{f}(d_n)(e)\right) = \bigsqcup_n f(d_n, e)$$

$$f\left(\bigsqcup_n d_n, e\right) = \bigsqcup_n f(d_n, e)$$

cont of f

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.

- <u>monotonicity</u>

$$g \sqsubseteq g' \text{ in } (E \to F) \wedge f \sqsubseteq f' \text{ in } (D \to E)$$

$$\overset{?}{\Longrightarrow}$$

$$g \circ f \sqsubseteq g' \circ f' \text{ in } (D \to F)$$

$$\Updownarrow$$

$$\text{iff } \forall d \in D.\ g(f(d)) \sqsubseteq g'(f'(d)) \text{ in } F$$

$$f \sqsubseteq f' \Rightarrow f(d) \sqsubseteq f'(d) \quad \forall d \in D$$

$$g \text{ monotone} \Rightarrow g(f(d)) \sqsubseteq g(f'(d))$$

$$g \sqsubseteq g' \Rightarrow g(f'(d)) \sqsubseteq g'(f'(d))$$

- <u>preservation of lubs</u>

for all $g_0 \sqsubseteq g_1 \sqsubseteq \cdots \sqsubseteq g_n \sqsubseteq \cdots \; (n \in \mathbb{N})$ in $(E \to F)$

and $f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \; (n \in \mathbb{N})$ in $(D \to E)$

$$\left(\bigsqcup_n g_n\right) \circ \left(\bigsqcup_n f_n\right) = \bigsqcup_n (g_n \circ f_n)$$

in $(D \to F)$

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

- <u>monotonicity</u>

$$f \sqsubseteq g \text{ in } (D \to D)$$

$$\Downarrow$$

$$\cfrac{\cfrac{\overline{f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g))} \qquad \overline{g(\mathit{fix}(g)) \sqsubseteq \mathit{fix}(g)}}{f(\mathit{fix}(g)) \sqsubseteq \mathit{fix}(g)}}{\underline{\mathit{fix}}(f) \sqsubseteq \underline{\mathit{fix}}(g)}$$

- <u>preservation</u> of <u>lubs</u>

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad (n \in \omega) \quad \text{in } (D \to D)$$

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_m \underline{fix}(f_m)\right) = \bigsqcup_n f_n\left(\bigsqcup_m \underline{fix}(f_m)\right)$$

$$= \bigsqcup_n \bigsqcup_m f_n(fix(f_m)) = \bigsqcup_k f_k(fix(f_k))$$

$$\frac{\forall k \quad f_k(fix\, f_k) \sqsubseteq fix(f_k)}{\bigsqcup_k f_k(fix(f_k))}$$

$$\|$$

$$\frac{\left(\bigsqcup_n f_n\right)\left(\bigsqcup_m fix(f_m)\right) \sqsubseteq \bigsqcup_m \underline{fix}(f_m)}{\underline{fix}\left(\bigsqcup_n f_n\right) \sqsubseteq \bigsqcup_n \underline{fix}(f_n)} \qquad \boxtimes$$