Topic 2

Least Fixed Points

Thesis

All domains of computation are partial orders with a least element.

$d \sqsubseteq d'$

$\Updownarrow$

- d approximates d'
- d' provides the same or more information than d

Thesis

All domains of computation are partial orders with a least element.

All computable functions are monotonic.

if $d \sqsubseteq d'$ then $f(d) \sqsubseteq f(d')$ ~ monotonicity.

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

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Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

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Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
 & & \forall x \in dom(f).\ f(x) = g(x) \\
 & \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Monotonicity

- A function $f : D \rightarrow E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Example: Sequencing a state transformer with itself.

$f : (State \to State) \to (State \to State)$

$f(w) = w \circ w = \lambda s \in State.\ w(w\,s).$

Monotonicity for f:

For $u, v \in State \to State$

if $u \sqsubseteq v$ then $f(u) \sqsubseteq f(v)$

So assume $u \sqsubseteq v$.

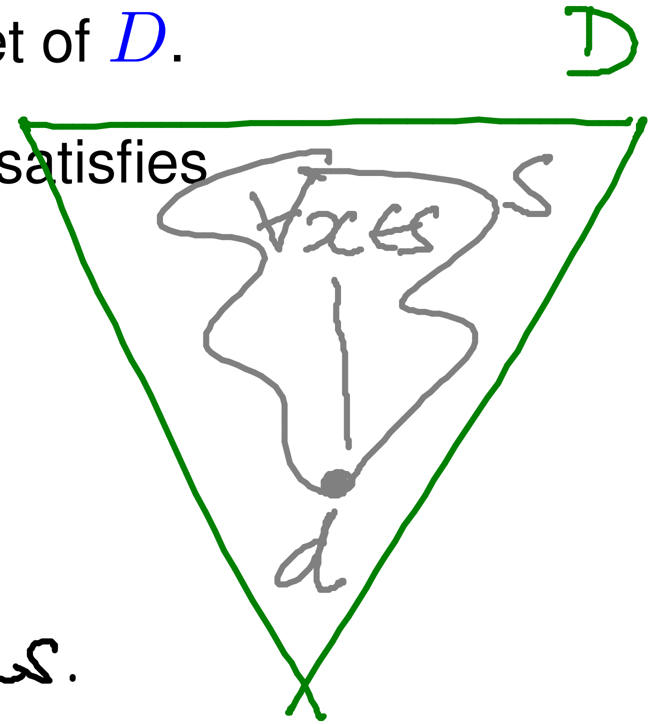
That is, for all states s, if $u(s)$ is defined then so is $v(s)$ and $u(s) = v(s)$.

We show $u^2 \sqsubseteq v^2$

That is, for all states s, if $u(us)$ is defined then so is $v(v(s))$ and they are equal. Indeed, suppose $u(us)$ is defined. Then $u(s)$ is defined and so is $v(s)$ with $u(s) = v(s)$. Also $v(us)$ is defined and $v(us) = u(us)$. So, $v(vs) = v(us) = u(us)$ is defined, and we are done. $\boxtimes$

Least Elements

Suppose that D is a poset and that S is a subset of D.



An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x \ .$$

Example:

The completely undefined function is least in the domain of partial functions.

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.

  s, s' are least in S then $s \sqsubseteq s'$ and $s' \sqsubseteq s$ so $s = s'$.

- Note also that a poset may not have least element.

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{fix(f)} = \text{least element of } \{d \in D \mid f(d) \sqsubseteq d\}$$

It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

Proof principle

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

Proof principle

1.
$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

$$\dfrac{\dfrac{}{f(\mathit{fix}\ f) \sqsubseteq \mathit{fix}\ f} \qquad \dfrac{*}{\mathit{fix}(f) \sqsubseteq f(\mathit{fix}\ f)}}{f(\mathit{fix}\ f) = \mathit{fix}(f)}$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

$$\dfrac{\dfrac{\dfrac{}{f(\mathit{fix}\ f) \sqsubseteq \mathit{fix}(f)}}{f(f(\mathit{fix}\ f)) \sqsubseteq f(\mathit{fix}\ f)}\ (f \text{ monotone})}{\mathit{fix}(f) \sqsubseteq f(\mathit{fix}(f))}$$

Thesis*

All domains of computation are
complete partial orders with a least element.

The information provided by a chain of elements in D

$$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad (n \in \mathbb{N})$$

can be <u>joined</u> as an element

$$\left(\bigsqcup_{n \in \mathbb{N}} d_n \right) \text{ in } D$$

Considering

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad (n \in \omega)$ in D

$$D \xrightarrow{f} E \quad \text{monotone}$$

Thesis*

All domains of computation are complete partial orders with a least element.

we will require that $f(\bigsqcup_n d_n)$ is obtained

All computable functions are continuous.

as the join of $f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots (n \in \omega)$

That is,

$$f\left(\bigsqcup_n d_n\right) = \bigsqcup_n f(d_n).$$

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

or join

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}$$

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

$$\frac{}{\bot \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{aligned}
f \sqsubseteq g \quad &\text{iff} \quad dom(f) \subseteq dom(g) \text{ and} \\
&\qquad \forall x \in dom(f).\ f(x) = g(x) \\
&\text{iff} \quad graph(f) \subseteq graph(g)
\end{aligned}
$$

✓

Lub of chain $\underline{f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots}$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$
f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}
$$

That is, $\underline{graph}(f) = \bigcup_{n \in \mathbb{N}} \underline{graphs}(f_n)$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\perp$ is the totally undefined partial function.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\dfrac{\forall k \qquad \dfrac{\overline{d_k \sqsubseteq e_k} \qquad \overline{e_k \sqsubseteq \bigsqcup_n e_n}}{d_k \sqsubseteq \bigsqcup_n e_n}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$
\dfrac{\dfrac{\checkmark}{d \sqsubseteq \bigsqcup_n d} \qquad \dfrac{\forall n \quad \dfrac{\checkmark}{d \sqsubseteq d}}{\bigsqcup_n d \sqsubseteq d}}{d = \bigsqcup_n d}
$$

$$\frac{①\ \bigsqcup_n d_n \sqsubseteq \bigsqcup_m d_{N+m} \qquad ②\ \bigsqcup_m d_{N+m} \sqsubseteq \bigsqcup_n d_n}{\bigsqcup_n d_n = \bigsqcup_m d_{N+m} \quad (N \in \mathbb{N})}$$

$$\frac{\dfrac{\dfrac{\checkmark}{d_n \sqsubseteq d_{N+n}} \qquad \dfrac{\checkmark}{d_{N+n} \sqsubseteq \bigsqcup_m d_{N+m}}}{\forall n \qquad d_n \sqsubseteq \bigsqcup_m d_{N+m}}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_m d_{N+m}}$$

$$\dfrac{\forall m \qquad \dfrac{\checkmark}{d_{N+m} \sqsubseteq \bigsqcup_n d_n}}{\bigsqcup_m d_{N+m} \sqsubseteq \bigsqcup_n d_n}$$

Consider a double chain: $d_{m,n}$ $(m, n \in \mathbb{N})$

s.t.

$\forall i<j,\ k<l.\ d_{i,k} \sqsubseteq d_{j,l}$

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & & & & & & & \\
\sqcup\!| & & \sqcup\!| & & & & & & & & \\
d_{0,n} & \sqsubseteq & d_{1,n} & \sqsubseteq & \cdots & & & & & & \\
\vdots & & \vdots & & & & & & \vdots & & \\
\sqcup\!| & & \sqcup\!| & & & & & & \sqcup\!| & & \\
d_{0,1} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{2,1} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,1} & \sqsubseteq & \cdots \\
\sqcup\!| & & \sqcup\!| & & \sqcup\!| & & & & \sqcup\!| & & \\
d_{0,0} & \sqsubseteq & d_{1,0} & \sqsubseteq & d_{2,0} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,0} & \sqsubseteq & \cdots
\end{array}$$

$$
\begin{array}{ccccccccccc}
\bigsqcup_n d_{0,n} & \sqsubseteq & \bigsqcup_n d_{1,n} & \sqsubseteq & \cdots & & & & & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & \\
\vdots & & \vdots & & & & & & & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & \\
d_{0,n} & \sqsubseteq & d_{1,n} & \sqsubseteq & \cdots & & & & & & \\
\vdots & & \vdots & & & & & & \vdots & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
d_{0,1} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{2,1} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,1} & \sqsubseteq & \cdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \\
d_{0,0} & \sqsubseteq & d_{1,0} & \sqsubseteq & d_{2,0} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,0} & \sqsubseteq & \cdots
\end{array}
$$

$$
\begin{array}{cccccccccccc}
\bigsqcup_n d_{0,n} & \sqsubseteq & \bigsqcup_n d_{1,n} & \sqsubseteq & \cdots & & & & & & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & & \\
\vdots & & \vdots & & & & & & & & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & & \\
d_{0,n} & \sqsubseteq & d_{1,n} & \sqsubseteq & \cdots & & & & & & & \\
\vdots & & \vdots & & & & & & \vdots & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,1} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{2,1} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,1} & \sqsubseteq & \cdots & \bigsqcup_m d_{m,1} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,0} & \sqsubseteq & d_{1,0} & \sqsubseteq & d_{2,0} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,0} & \sqsubseteq & \cdots & \sqsubseteq \bigsqcup_m d_{m,0}
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
\bigsqcup_n d_{0,n} & \sqsubseteq & \bigsqcup_n d_{1,n} & \sqsubseteq & \cdots & & & & \sqsubseteq & \bigsqcup_m \bigsqcup_n d_{m,n} & & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & & & \\
\vdots & & \vdots & & & & & & & & & & \bigsqcup_n \bigsqcup_m d_{m,n} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,n} & \sqsubseteq & d_{1,n} & \sqsubseteq & \cdots & & & & & & & & \\
\vdots & & \vdots & & & & & & \vdots & & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,1} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{2,1} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,1} & \sqsubseteq & \cdots & & \bigsqcup_m d_{m,1} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,0} & \sqsubseteq & d_{1,0} & \sqsubseteq & d_{2,0} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,0} & \sqsubseteq & \cdots & \sqsubseteq & \bigsqcup_m d_{m,0}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
\bigsqcup_n d_{0,n} & \sqsubseteq & \bigsqcup_n d_{1,n} & \sqsubseteq & \cdots & & & \sqsubseteq & \bigsqcup_m \bigsqcup_n d_{m,n} & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \| \text{ lemma} & & \\
\vdots & & \vdots & & & & & & \bigsqcup_k d_{k,k} & = & \bigsqcup_n \bigsqcup_m d_{m,n} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \unicode{x22F0} & & & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,n} & \sqsubseteq & d_{1,n} & \sqsubseteq & \cdots & d_{k,k} & & & & & \\
\vdots & & \vdots & & \unicode{x22F0} & & & & \vdots & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,1} & \sqsubseteq & d_{1,1} & \sqsubseteq & d_{2,1} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,1} & \sqsubseteq \cdots & \bigsqcup_m d_{m,1} \\
\rotatebox{90}{$\sqsubseteq$} & \unicode{x22F0} & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\
d_{0,0} & \sqsubseteq & d_{1,0} & \sqsubseteq & d_{2,0} & \sqsubseteq & \cdots & \sqsubseteq & d_{m,0} & \sqsubseteq \cdots & \sqsubseteq \bigsqcup_m d_{m,0}
\end{array}
$$

$$
\dfrac{\dfrac{\dfrac{\checkmark}{d_{k,k} \sqsubseteq \bigsqcup_n d_{k,n}} \qquad \dfrac{\checkmark}{\bigsqcup_n d_{k,n} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}}{\forall k \qquad d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}}{\bigsqcup_k d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}
$$

$$l =^{\mathrm{def}} \max(m,n)$$

$$
\cfrac{
  \forall m \quad \cfrac{
    \forall n \quad \cfrac{
      \cfrac{\checkmark}{d_{m,n} \sqsubseteq d_{l,l}}
      \qquad
      \cfrac{\checkmark}{d_{l,l} \sqsubseteq \bigsqcup_k d_{k,k}}
    }{
      d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}
    }
  }{
    \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}
  }
}{
  \bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}
}
$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \ \& \ n \leq n' \ \Rightarrow \ d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n\geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,3} \sqsubseteq \ldots$$

Moreover

$$\bigsqcup_{m\geq 0}\left(\bigsqcup_{n\geq 0} d_{m,n}\right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0}\left(\bigsqcup_{m\geq 0} d_{m,n}\right) .$$

$\triangleright$ $f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n f(d_n)$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E. \checkmark$$

Note

$$\frac{d_n \sqsubseteq \bigsqcup_n d_n}{\forall n \quad f(d_n) \sqsubseteq f(\bigsqcup_n d_n)} \; f \text{ monotone}$$

$$\frac{\forall n \quad f(d_n) \sqsubseteq f(\bigsqcup_n d_n)}{\bigsqcup_n f(d_n) \sqsubseteq f(\bigsqcup_n d_n)}$$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\bot) = \bot$.

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

PROOF OF TARSKI'S FIXED POINT THEOREM

(1) $\perp \sqsubseteq f(\perp) \xRightarrow[\text{f monotone}]{} f^n(\perp) \sqsubseteq f^{n+1}(\perp)$

$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots$$

$(n \in \mathbb{N})$

$$\bigsqcup_n f^n(\perp) = \underline{fix}(f)$$

(2) $\mathrm{fix}(f)$ is a fixed point

$$f\left(\bigsqcup_n f^n(\perp)\right)$$

$\parallel$ $\quad$ f continuous

$$\bigsqcup_n f(f^n(\perp)) = \bigsqcup\left(f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots\right)$$

$$= \bigsqcup\left(\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots\right)$$

$$= \bigsqcup_n f^n(\perp).$$

(3) $\mathrm{fix}(f)$ is least amongst prefixed points.

Let d be a prefixed point, i.e. $f(d) \sqsubseteq d$.

Then, it follows by induction using that f is monotone and that $\bot$ is a least element, that $\forall n \in \mathbb{N}.\ f^n(\bot) \sqsubseteq d$.

Hence, $\bigsqcup_n f^n \bot \sqsubseteq d$. $\boxtimes$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$ $\quad f_{[\![B]\!],[\![C]\!]} : (State \rightharpoonup State) \to (State \rightharpoonup State)$

$= \bigsqcup_{n \geq 0} {f_{[\![B]\!],[\![C]\!]}}^n(\bot)$ $\quad$ CONTINUOUS!

$$= \lambda s \in State. \begin{cases} [\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$