Complexity Theory

Lecture 11

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http://www.cl.cam.ac.uk/teaching/1920/Complexity

Logarithmic Space Reductions

We write

$$A \leq_L B$$

if there is a reduction f of A to B that is computable by a deterministic Turing machine using $O(\log n)$ workspace (with a *read-only* input tape and *write-only* output tape).

Note: We can compose $\leq_L$ reductions. So,

$$\text{if } A \leq_L B \text{ and } B \leq_L C \text{ then } A \leq_L C$$

NP-complete Problems

Analysing carefully the reductions we constructed in our proofs of NP-completeness, we can see that SAT and the various other NP-complete problems are actually complete under $\leq_L$ reductions.

Thus, if SAT $\leq_L A$ for some problem A in L then not only P $=$ NP but also L $=$ NP.

P-complete Problems

It makes little sense to talk of complete problems for the class P with respect to polynomial time reducibility $\leq_P$.

There are problems that are complete for P with respect to *logarithmic space* reductions $\leq_L$.
One example is CVP—the circuit value problem.

That is, for every language A in P,

$$A \leq_L \mathsf{CVP}$$

- If $\mathsf{CVP} \in \mathsf{L}$ then $\mathsf{L} = \mathsf{P}$.
- If $\mathsf{CVP} \in \mathsf{NL}$ then $\mathsf{NL} = \mathsf{P}$.

Circuits

A circuit is a directed graph $G = (V, E)$, with $V = \{1, \ldots, n\}$ together with a labeling: $l : V \rightarrow \{\mathtt{true}, \mathtt{false}, \wedge, \vee, \neg\}$, satisfying:

- If there is an edge (i, j), then $i < j$;
- Every node in V has *indegree* at most 2.
- A node v has
  indegree 0 iff $l(v) \in \{\mathtt{true}, \mathtt{false}\}$;
  indegree 1 iff $l(v) = \neg$;
  indegree 2 iff $l(v) \in \{\vee, \wedge\}$

The value of the expression is given by the value at node n.

CVP

A circuit is a more compact way of representing a Boolean expression.

Identical subexpressions need not be repeated.

CVP - the *circuit value problem* is, given a circuit, determine the value of the result node n.

CVP is solvable in polynomial time, by the algorithm which examines the nodes in increasing order, assigning a value `true` or `false` to each node.

Reachability

Similarly, it can be shown that Reachability is, in fact, NL-complete.

For any language $A \in \mathsf{NL}$, we have $A \leq_L$ Reachability

$\mathsf{L} = \mathsf{NL}$ *if, and only if,* *Reachability* $\in \mathsf{L}$

Note: it is known that the reachability problem for *undirected* graphs is in L.

Provable Intractability

Our aim now is to show that there are languages (*or, equivalently, decision problems*) that we can prove are not in P.

This is done by showing that, for every *reasonable* function f, there is a language that is not in $\mathsf{TIME}(f)$.

The proof is based on the diagonal method, as in the proof of the undecidability of the halting problem.

Time Hierarchy Theorem

For any constructible function f, with $f(n) \geq n$, define the f-bounded *halting language* to be:

$$H_f = \{[M], x \mid M \text{ accepts } x \text{ in } f(|x|) \text{ steps}\}$$

where $[M]$ is a description of M in some fixed encoding scheme.
Then, we can show
$H_f \in \mathsf{TIME}(f(n)^2)$ and $H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

Time Hierarchy Theorem
For any constructible function $f(n) \geq n$, $\mathsf{TIME}(f(n))$ is properly contained in $\mathsf{TIME}(f(2n+1)^2)$.

Strong Hierarchy Theorems

For any constructible function $f(n) \geq n$, $\mathsf{TIME}(f(n))$ is properly contained in $\mathsf{TIME}(f(n)(\log f(n)))$.

Space Hierarchy Theorem
For any pair of constructible functions f and g, with $f = O(g)$ and $g \neq O(f)$, there is a language in $\mathsf{SPACE}(g(n))$ that is not in $\mathsf{SPACE}(f(n))$.

Similar results can be established for nondeterministic time and space classes.

Consequences

- For each k, $\mathsf{TIME}(n^k) \neq \mathsf{P}$.
- $\mathsf{P} \neq \mathsf{EXP}$.
- $\mathsf{L} \neq \mathsf{PSPACE}$.
- Any language that is EXP-complete is not in P.
- There are no problems in P that are complete under linear time reductions.