Lecture 15

Presheaf categories

Let C be a small category. The functor category Set^{C°P} is called the category of preseaves on C.

- objects are contravariant functors from C to Set
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.

$$y: \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}$$

(where C is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

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For each C-object X, the object $yX \in Set^{C^{op}}$ is the functor $C(_-,X):C^{op} \to Set$ given by

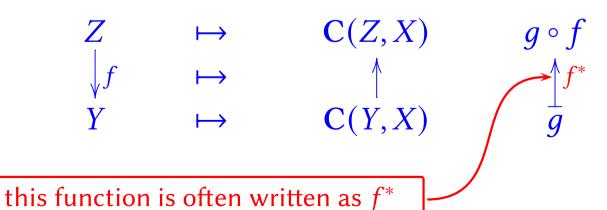
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For each C-morphism $Y \xrightarrow{f} X$, the morphism $yY \xrightarrow{yf} yX$ in $Set^{C^{op}}$ is the natural transformation whose component at any given $Z \in C^{op}$ is the function

$$yY(Z) \xrightarrow{(yf)_Z} yX(Z)$$

$$C(Z, Y) \qquad C(Z, X)$$

$$g \longmapsto f \circ g$$

$$y: C \rightarrow Set^{C^{op}}$$

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 $yY(Z) \xrightarrow{(yf)_Z} yX(Z)$ $U = C(Z,Y) \qquad C(Z,X)$ $written as <math>f_*$ $Q \longmapsto f_* \qquad f \circ g$

For each small category \mathbb{C} , each object $X \in \mathbb{C}$ and each presheaf $F \in \operatorname{Set}^{\mathbb{C}^{op}}$, there is a bijection of sets

$$\eta_{X,F}: \mathbf{Set}^{\mathbf{C^{op}}}(yX,F) \cong F(X)$$

which is natural in both X and F.

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the value of $F: \mathbb{C}^{op} \to \mathbf{Set}$ at X

the set of natural transformations from the functor $yX : \mathbb{C}^{op} \to \mathbf{Set}$

to the functor $F: \mathbb{C}^{op} \to \mathbf{Set}$

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which is natural in both X and F.

Definition of the function $\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}(yX,F) \to F(X)$:

for each $\theta : yX \to F$ in $\mathbf{Set}^{\mathbf{C^{op}}}$ we have the function $\mathbf{C}(X,X) = yX(X) \xrightarrow{\theta_X} F(X)$ and define

 $\eta_{X,F}(\theta) \triangleq \theta_X(\mathrm{id}_X)$

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which is natural in both X and F.

Definition of the function $\eta_{X,F}^{-1}: F(X) \to \operatorname{Set}^{\operatorname{C^{op}}}(yX,F):$ for each $x \in F(X), Y \in \mathbb{C}$ and $f \in yX(Y) = \mathbb{C}(Y,X),$ we get a $F(X) \xrightarrow{F(f)} F(Y)$ in Set and hence $F(f)(x) \in F(Y);$

For each small category C, each object $X \in C$ and each presheaf $F \in Set^{C^{op}}$, there is a bijection of sets

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for each $x \in F(X)$, $Y \in \mathbb{C}$ and $f \in yX(Y) = \mathbb{C}(Y,X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in **Set** and hence $F(f)(x) \in F(Y)$;

Define
$$\left(\eta_{X,F}^{-1}(x)\right)_Y: yX(Y) \to F(Y)$$
 by

$$\left(\eta_{X,F}^{-1}(x)\right)_{Y}(f) \triangleq F(f)(x)$$

check this gives a natural transformation $\eta_{X,F}^{-1}(x): yX \to F$

Proof of
$$\eta_{X,F} \circ \eta_{X,F}^{-1} = id_{F(X)}$$

For any $x \in F(X)$ we have

$$\eta_{X,F} \left(\eta_{X,F}^{-1}(x) \right) \triangleq \left(\eta_{X,F}^{-1}(x) \right)_{X} (id_{X})$$

$$\triangleq F(id_{X})(x)$$

$$= id_{F(X)}(x)$$

$$= x$$

by definition of $\eta_{X,F}$ by definition of $\eta_{X,F}^{-1}$ since F is a functor

Proof of
$$\eta_{X,F}^{-1} \circ \eta_{X,F} = id_{Set^{C^{op}}(yX,F)}$$

For any $yX \xrightarrow{\theta} F$ in $Set^{C^{op}}$ and $Y \xrightarrow{f} X$ in C, we have

$$\left(\eta_{X,F}^{-1} \left(\eta_{X,F}(\theta) \right) \right)_{Y} f \triangleq \left(\eta_{X,F}^{-1} \left(\theta_{X}(\mathrm{id}_{X}) \right) \right)_{Y} f$$

$$\triangleq F(f) (\theta_{X}(\mathrm{id}_{X}))$$

$$\triangleq \theta_{Y}(f^{*}(id_{X}))$$

$$\triangleq \theta_{Y}(\mathrm{id}_{X} \circ f)$$

$$= \theta_{Y}(f)$$

$$\text{naturality of } \theta$$

$$yX(Y) \xrightarrow{\theta_{Y}} F(Y)$$

$$\uparrow_{f^{*}} \qquad \uparrow_{F(f)}$$

$$yX(X) \xrightarrow{\theta_{X}} F(X)$$

by definition of $\eta_{X,F}$ by definition of $\eta_{X,F}^{-1}$ by naturality of θ by definition of f^*

Proof of
$$\eta_{X,F}^{-1} \circ \eta_{X,F} = id_{Set^{C^{op}}(yX,F)}$$

For any $yX \xrightarrow{\theta} F$ in $Set^{C^{op}}$ and $Y \xrightarrow{f} X$ in C, we have

$$\left(\eta_{X,F}^{-1} \left(\eta_{X,F}(\theta) \right) \right)_{Y} f \triangleq \left(\eta_{X,F}^{-1} \left(\theta_{X}(\mathrm{id}_{X}) \right) \right)_{Y} f$$

$$\triangleq F(f) (\theta_{X}(\mathrm{id}_{X}))$$

$$= \theta_{Y} (f^{*}(id_{X}))$$

$$\triangleq \theta_{Y}(\mathrm{id}_{X} \circ f)$$

$$= \theta_{Y}(f)$$

by definition of $\eta_{X,F}$ by definition of $\eta_{X,F}^{-1}$ by naturality of θ by definition of f^*

so
$$\forall \theta, Y, \left(\eta_{X,F}^{-1}\left(\eta_{X,F}(\theta)\right)\right)_{Y} = \theta_{Y}$$

so $\forall \theta, \ \eta_{X,F}^{-1}\left(\eta_{X,F}(\theta)\right) = \theta$
so $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}.$

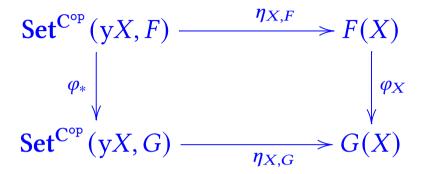
For each small category C, each object $X \in C$ and each presheaf $F \in \mathbf{Set}^{C^{op}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}(yX, F) \cong F(X)$$

which is natural in both *X* and *F*.

Proof that $\eta_{X,F}$ is natural in F:

Given $F \xrightarrow{\varphi} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, have to show that



commutes in Set. For all $yX \xrightarrow{\theta} F$ we have

$$\varphi_X (\eta_{X,F}(\theta)) \triangleq \varphi_X (\theta_X(\mathrm{id}_X))$$

$$\triangleq (\varphi \circ \theta)_X(\mathrm{id}_X)$$

$$\triangleq \eta_{X,G}(\varphi \circ \theta)$$

$$\triangleq \eta_{X,G}(\varphi_*(\theta))$$

Proof that $\eta_{X,F}$ is natural in X:

Given $Y \xrightarrow{f} X$ in \mathbb{C} , have to show that

$$\begin{array}{c|c}
\operatorname{Set}^{C^{\operatorname{op}}}(yX, F) & \xrightarrow{\eta_{X,F}} & F(X) \\
(yf)^* & & & \downarrow^{F(f)} \\
\operatorname{Set}^{C^{\operatorname{op}}}(yY, F) & \xrightarrow{\eta_{Y,F}} & F(Y)
\end{array}$$

commutes in Set. For all $yX \xrightarrow{\theta} F$ we have

$$F(f)((\eta_{X,F}(\theta)) \triangleq F(f)(\theta_X(\mathrm{id}_X))$$

$$= \theta_Y(f^*(\mathrm{id}_X)) \qquad \text{by naturality of } \theta$$

$$= \theta_Y(f)$$

$$= \theta_Y(f_*(\mathrm{id}_Y))$$

$$\triangleq (\theta \circ yf)_Y(\mathrm{id}_Y)$$

$$\triangleq \eta_{Y,F}(\theta \circ yf)$$

$$\triangleq \eta_{Y,F}((yf)^*(\theta))$$

Corollary of the Yoneda Lemma:

the functor $y: C \to Set^{C^{op}}$ is full and faithful.

In general, a functor $F: \mathbb{C} \to \mathbb{D}$ is

▶ faithful if for all $X, Y \in \mathbb{C}$ the function

$$\mathbf{C}(X,Y) \rightarrow \mathbf{D}(F(X),F(Y))$$
 $f \mapsto F(f)$

is injective:

$$\forall f, f' \in \mathbf{C}(X, Y), \ F(f) = F(f') \Rightarrow f = f'$$

full if the above functions are all surjective:

$$\forall g \in \mathbf{D}(F(X), F(Y)), \exists f \in \mathbf{C}(X, Y), F(f) = g$$

Corollary of the Yoneda Lemma:

the functor $y: C \to Set^{C^{op}}$ is full and faithful.

Proof. From the proof of the Yoneda Lemma, for each $F \in \mathbf{Set}^{\mathbf{C}^{op}}$ we have a bijection

$$F(X) \xrightarrow{(\eta_{X,F})^{-1}} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(yX,F)$$

By definition of $(\eta_{X,F})^{-1}$, when F = yY the above function is equal to

$$yY(X) = C(X, Y) \rightarrow Set^{C^{op}}(yX, yY)$$

 $f \mapsto f_* = yf$

So, being a bijection, $f \mapsto yf$ is both injective and surjective; so y is both faithful and full.

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Recall (for a small category C):

Yoneda functor $y : \mathbb{C} \to Set^{\mathbb{C}^{op}}$

Yoneda Lemma: there is a bijection $Set^{C^{op}}(yX, F) \cong F(X)$ which is natural both in $F \in Set^{C^{op}}$ and $X \in \mathbb{C}$.

An application of the Yoneda Lemma:

Theorem. For each small category C, the category Set^{Cop} of presheaves is cartesian closed.

Proof sketch.

Terminal object in $\mathbf{Set}^{\mathbf{C}^{op}}$ is the functor $\mathbf{1}: \mathbf{C}^{op} \to \mathbf{Set}$ given by

$$\begin{cases} 1(X) \triangleq \{0\} & \text{terminal object in Set} \\ 1(f) \triangleq \text{id}_{\{0\}} \end{cases}$$

Proof sketch.

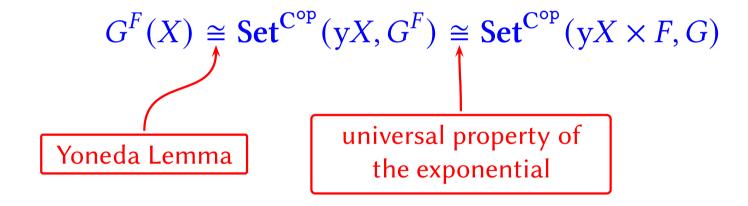
Product of $F, G \in \mathbf{Set}^{\mathbf{C^{op}}}$ is the functor $F \times G : \mathbf{C^{op}} \to \mathbf{Set}$ given by

$$\begin{cases} (F \times G)(X) \triangleq F(X) \times G(X) & \text{cartesian product of sets} \\ (F \times G)(f) \triangleq F(f) \times G(f) \end{cases}$$

with projection morphisms $F \stackrel{\pi_1}{\longleftarrow} F \times G \stackrel{\pi_2}{\longrightarrow} G$ given by the natural transformations whose components at $X \in \mathbb{C}$ are the projection functions $F(X) \stackrel{\pi_1}{\longleftarrow} F(X) \times G(X) \stackrel{\pi_2}{\longrightarrow} G(X)$.

Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:



Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{op}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \cong \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}(yX, G^F) \cong \mathbf{Set}^{\mathbf{C}^{\mathsf{op}}}(yX \times F, G)$$

We take the set $Set^{C^{op}}(yX \times F, G)$ to be the definition of the value of G^F at X...

Exponential objects in Set^{Cop}:

$$G^F(X) \triangleq \mathbf{Set}^{\mathbf{C^{op}}}(yX \times F, G)$$

Given $Y \xrightarrow{f} X$ in \mathbb{C} , we have $yY \xrightarrow{yf} yX$ in $Set^{\mathbb{C}^{op}}$ and hence

$$G^{F}(Y) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(yY \times F, G) \rightarrow \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(yX \times F, G) \triangleq G^{F}(X)$$

 $\theta \mapsto \theta \circ (yf \times id_{F})$

We define

$$G^F(f) \triangleq (yf \times id_F)^*$$

Have to check that these definitions make G^F into a functor $C^{op} \rightarrow Set$.

Application morphisms in Set^{Cop}:

Given $F, G \in \mathbf{Set}^{\mathbf{C^{op}}}$, the application morphism

$$app: G^F \times F \to G$$

is the natural transformation whose component at $X \in \mathbb{C}$ is the function

$$(G^F \times F)(X) \triangleq G^F(X) \times F(X) \triangleq \mathbf{Set}^{\mathsf{C}^{\mathsf{op}}}(yX \times F, G) \times F(X) \xrightarrow{\mathsf{app}_X} G(X)$$

defined by

$$app_X(\theta, x) \triangleq \theta_X(id_X, x)$$

Have to check that this is natural in X.

Currying operation in Set^{Cop}:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\operatorname{cur} \theta} G^F\right)$$

Given $H \times F \xrightarrow{\theta} G$ in $Set^{C^{op}}$, the component of $cur \theta$ at $X \in C$

$$H(X) \xrightarrow{(\operatorname{cur} \theta)_X} G^F(X) \triangleq \operatorname{Set}^{\operatorname{C^{op}}}(yX \times F, G)$$

is the function mapping each $z \in H(X)$ to the natural transformation $yX \times F \to G$ whose component at $Y \in \mathbb{C}$ is the function

$$(yX \times F)(Y) \triangleq \mathbf{C}(Y,X) \times F(Y) \rightarrow G(Y)$$

defined by

$$((\operatorname{cur} \theta)_X(z))_Y(f,y) \triangleq \theta_Y(H(f)(z),y)$$

Currying operation in Set^{Cop}:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\operatorname{cur} \theta} G^F\right)$$

$$((\operatorname{cur} \theta)_X(z))_Y(f,y) \triangleq \theta_Y(H(f)(z),y)$$

Have to check that this is natural in Y,

then that $(\operatorname{cur} \theta)_X$ is natural in X,

then that $\operatorname{cur} \theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ in $\operatorname{Set}^{C^{\circ p}}$ satisfying $\operatorname{app} \circ (\varphi \times \operatorname{id}_F) = \theta$.

So we can interpret simply typed lambda calculus in any presheaf category.

More than that, presheaf categories (usefully) model dependently-typed languages.