Lecture 15
Presheaf categories

Let $\mathbf{C}$ be a small category. The functor category $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ is called the category of preseaves on $\mathbf{C}$.

- objects are contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.
Yoneda functor

\[ y : C \rightarrow \text{Set}^{\text{C}^{\text{op}}} \]

(where \( C \) is a small category)

is the Curried version of the \( \text{hom} \) functor

\[ C \times \text{C}^{\text{op}} \cong \text{C}^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set} \]
Yoneda functor

\[ y : C \to \text{Set}^{C^\text{op}} \]

(where \( C \) is a small category)

is the Curried version of the \textit{hom} functor

\[ C \times C^\text{op} \cong C^\text{op} \times C \overset{\text{Hom}_C}{\longrightarrow} \text{Set} \]

- For each \( C \)-object \( X \), the object \( yX \in \text{Set}^{C^\text{op}} \) is the functor \( C(\_, X) : C^\text{op} \to \text{Set} \) given by

\[
\begin{array}{ccc}
Z & \mapsto & C(Z, X) \\
\downarrow^f & & \uparrow \\
Y & \mapsto & C(Y, X)
\end{array}
\]

\[ g \circ f \]
Yoneda functor

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(where \(C\) is a small category)

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• For each \(C\)-object \(X\), the object \(yX \in \text{Set}^{C^{\text{op}}}\) is the functor \(C(\_ , X) : C^{\text{op}} \to \text{Set}\) given by

\[
\begin{align*}
Z & \mapsto \quad C(Z,X) \\
\downarrow f & \mapsto \quad \uparrow g \circ f \\
Y & \mapsto \quad C(Y,X)
\end{align*}
\]

this function is often written as \(f^*\)
Yoneda functor

\[ y : C \to \text{Set}^{\text{op}} \]  

(where \( C \) is a small category)

is the Curried version of the hom functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \overset{\text{Hom}_C}{\longrightarrow} \text{Set} \]

\[ \triangleright \text{For each } C\text{-morphism } Y \xrightarrow{f} X, \text{ the morphism } yY \xrightarrow{yf} yX \text{ in } \text{Set}^{\text{op}} \text{ is the natural transformation whose component at any given } Z \in C^{\text{op}} \text{ is the function} \]

\[
\begin{align*}
yY(Z) \xrightarrow{(yf)_Z} yX(Z) \\
\| \\
C(Z, Y) \xrightarrow{g} C(Z, X)
\end{align*}
\]

\[ f \circ g \]
Yoneda functor

\[ y : C \to \text{Set}^{\text{op}} \]

(where \( C \) is a small category)

is the Curried version of the \( \text{hom} \) functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \overset{\text{Hom}_C}{\longrightarrow} \text{Set} \]

- For each \( C \)-morphism \( Y \xrightarrow{f} X \), the morphism \( yY \xrightarrow{yf} yX \) in \( \text{Set}^{\text{op}} \) is the natural transformation whose component at any given \( Z \in C^{\text{op}} \) is the function

\[ \begin{align*}
yY(Z) \xrightarrow{(yf)_Z} yX(Z) \\
\| & \| \\
\text{C}(Z,Y) & \xrightarrow{f_*} \text{C}(Z,X)
\end{align*} \]

this function is often written as \( f_* \)
The Yoneda Lemma

For each small category $C$, each object $X \in C$ and each presheaf $F \in \text{Set}^{C^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{C^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$. 

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The Yoneda Lemma

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which is natural in both $X$ and $F$. 

- the set of natural transformations from the functor $yX : C^\text{op} \to \text{Set}$ to the functor $F : C^\text{op} \to \text{Set}$
- the value of $F : C^\text{op} \to \text{Set}$ at $X$
The Yoneda Lemma

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$$\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \to F(X)$:

For each $\theta : yX \to F$ in $\text{Set}^{\mathbf{C}^{\text{op}}}$ we have the function $\mathbf{C}(X, X) = yX(X) \xrightarrow{\theta_X} F(X)$ and define

$$\eta_{X,F}(\theta) \triangleq \theta_X(\text{id}_X)$$
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \text{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}} (yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F}^{-1} : F(X) \to \text{Set}^{\mathbf{C}^{\text{op}}} (yX, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in yX(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\text{Set}$ and hence $F(f)(x) \in F(Y)$;
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

**Definition of the function** $\eta_{X,F}^{-1} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in yX(Y) = \mathbf{C}(Y, X)$, we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;

Define $\left( \eta_{X,F}^{-1}(x) \right)_Y : yX(Y) \to F(Y)$ by

$$\left( \eta_{X,F}^{-1}(x) \right)_Y (f) \triangleq F(f)(x)$$

check this gives a natural transformation $\eta_{X,F}(x) : yX \to F$
Proof of $\eta_{X,F} \circ \eta_{X,F}^{-1} = \text{id}_{F(X)}$

For any $x \in F(X)$ we have

$$\eta_{X,F} \left( \eta_{X,F}^{-1}(x) \right) \triangleq \left( \eta_{X,F}^{-1}(x) \right)_X (\text{id}_X)$$

by definition of $\eta_{X,F}$

$$\triangleq F(\text{id}_X)(x)$$

by definition of $\eta_{X,F}^{-1}$

$$= \text{id}_{F(X)}(x)$$

since $F$ is a functor

$$= x$$
Proof of $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}_{\text{Set}^{\text{op}}(yX,F)}$

For any $yX \xrightarrow{\theta} F$ in $\text{Set}^{\text{op}}$ and $Y \xrightarrow{f} X$ in $\mathcal{C}$, we have

$$\left(\eta_{X,F}^{-1} \eta_{X,F}(\theta)\right)_Y f \overset{\Delta}{=} \left(\eta_{X,F}^{-1} (\theta_X(\text{id}_X))\right)_Y f$$

by definition of $\eta_{X,F}$

$$\overset{\Delta}{=} F(f)(\theta_X(\text{id}_X))$$

by definition of $\eta_{X,F}^{-1}$

$$= \theta_Y(f^*(\text{id}_X))$$

by naturality of $\theta$

$$\overset{\Delta}{=} \theta_Y(\text{id}_X \circ f)$$

by definition of $f^*$

$$= \theta_Y(f)$$

by naturality of $\theta$

\[\text{Diagram:}\]

\begin{align*}
yX(Y) & \xrightarrow{\theta_Y} F(Y) \\
yX(X) & \xrightarrow{\theta_X} F(X) \\
f^* & \\
F(f) &
\end{align*}
Proof of $\eta_{X,F}^{-1} \circ \eta_{X,F} = id_{Set^{\text{op}}(yX,F)}$

For any $yX \overset{\theta}{\to} F$ in $\text{Set}^{\text{op}}$ and $Y \overset{f}{\to} X$ in $C$, we have

$$\left(\eta_{X,F}^{-1} \left(\eta_{X,F}(\theta)\right)\right)_Y f \triangleq \left(\eta_{X,F}^{-1} \left(\theta_X(id_X)\right)\right)_Y f$$

by definition of $\eta_{X,F}$

$$\triangleq F(f)(\theta_X(id_X))$$

by definition of $\eta_{X,F}^{-1}$

$$= \theta_Y(f^*(id_X))$$

by naturality of $\theta$

$$\triangleq \theta_Y(id_X \circ f)$$

by definition of $f^*$

$$= \theta_Y(f)$$

so $\forall \theta, Y, \left(\eta_{X,F}^{-1} \left(\eta_{X,F}(\theta)\right)\right)_Y = \theta_Y$

so $\forall \theta, \eta_{X,F}^{-1} \left(\eta_{X,F}(\theta)\right) = \theta$

so $\eta_{X,F}^{-1} \circ \eta_{X,F} = id$. 
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \text{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}} (yX, F) \cong F(X)$$

which is natural in both $X$ and $F$. 
Proof that $\eta_{X,F}$ is natural in $F$:

Given $F \xrightarrow{\varphi} G$ in $\text{Set}^{\text{op}}$, have to show that

$\text{Set}^{\text{op}}(yX, F) \xrightarrow{\eta_{X,F}} F(X)$

$\text{Set}^{\text{op}}(yX, G) \xrightarrow{\eta_{X,G}} G(X)$

commutes in $\text{Set}$. For all $yX \xrightarrow{\theta} F$ we have

$\varphi_X (\eta_{X,F}(\theta)) \triangleq \varphi_X (\theta_X(id_X))$

$\triangleq (\varphi \circ \theta)_X(id_X)$

$\triangleq \eta_{X,G}(\varphi \circ \theta)$

$\triangleq \eta_{X,G}(\varphi_* (\theta))$
Proof that $\eta_{X,F}$ is natural in $X$:

Given $Y \xrightarrow{f} X$ in $C$, have to show that

$$\text{Set}^\text{C}^\text{op}(yX, F) \xrightarrow{\eta_{X,F}} F(X)$$

$$\text{Set}^\text{C}^\text{op}(yY, F) \xrightarrow{\eta_{Y,F}} F(Y)$$

commutes in $\text{Set}$. For all $yX \xrightarrow{\theta} F$ we have

$$F(f)((\eta_{X,F}(\theta)) \triangleq F(f)(\theta_Y(id_X))$$

$$= \theta_Y(f^*(id_X))$$

$$= \theta_Y(f)$$

$$= \theta_Y(f^*(id_Y))$$

$$\triangleq (\theta \circ yf)_Y(id_Y)$$

$$\triangleq \eta_{Y,F}(\theta \circ yf)$$

$$\triangleq \eta_{Y,F}((yf)^*(\theta))$$

by naturality of $\theta$
Corollary of the Yoneda Lemma:

the functor \( y : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}} \) is full and faithful.

In general, a functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) is

- faithful if for all \( X, Y \in \mathcal{C} \) the function
  
  \[
  \mathcal{C}(X, Y) \rightarrow \mathcal{D}(F(X), F(Y))
  \]
  
  \[ f \mapsto F(f) \]

  is injective:
  
  \[
  \forall f, f' \in \mathcal{C}(X, Y), \ F(f) = F(f') \Rightarrow f = f' \]

- full if the above functions are all surjective:
  
  \[
  \forall g \in \mathcal{D}(F(X), F(Y)), \exists f \in \mathcal{C}(X, Y), \ F(f) = g \]
Corollary of the Yoneda Lemma:

the functor \( y : C \to \text{Set}^{\text{Cop}} \) is full and faithful.

**Proof.** From the proof of the Yoneda Lemma, for each \( F \in \text{Set}^{\text{Cop}} \) we have a bijection

\[
F(X) \xrightarrow{(\eta_{X,F})^{-1}} \text{Set}^{\text{Cop}}(yX, F)
\]

By definition of \((\eta_{X,F})^{-1}\), when \( F = yY \) the above function is equal to

\[
yY(X) = C(X, Y) \rightarrow \text{Set}^{\text{Cop}}(yX, yY)
f \mapsto f_* = yf
\]

So, being a bijection, \( f \mapsto yf \) is both injective and surjective; so \( y \) is both faithful and full. \( \Box \)
Recall (for a small category $C$):

**Yoneda functor** $y : C \to \text{Set}^{\text{C}^{\text{op}}}$

**Yoneda Lemma:** there is a bijection $\text{Set}^{\text{C}^{\text{op}}}(yX, F) \cong F(X)$ which is natural both in $F \in \text{Set}^{\text{C}^{\text{op}}}$ and $X \in C$.

An application of the Yoneda Lemma:

**Theorem.** For each small category $C$, the category $\text{Set}^{\text{C}^{\text{op}}}$ of presheaves is cartesian closed.
Theorem. For each small category $\mathcal{C}$, the category $\text{Set}^{\mathcal{C}^{\text{op}}}$ of presheaves is cartesian closed.
Theorem. For each small category $C$, the category $\text{Set}^{C^{\text{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Terminal object in $\text{Set}^{C^{\text{op}}}$ is the functor $1 : C^{\text{op}} \rightarrow \text{Set}$ given by

\[
\begin{aligned}
1(X) &\triangleq \{0\} \quad \text{terminal object in } \text{Set} \\
1(f) &\triangleq \text{id}_{\{0\}}
\end{aligned}
\]
Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Product of $F, G \in \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ is the functor $F \times G : \mathbf{C}^{\text{op}} \to \mathbf{Set}$ given by

\[
\begin{align*}
(F \times G)(X) & \triangleq F(X) \times G(X) \quad \text{cartesian product of sets} \\
(F \times G)(f) & \triangleq F(f) \times G(f)
\end{align*}
\]

with projection morphisms $F \xleftarrow{\pi_1} F \times G \xrightarrow{\pi_2} G$ given by the natural transformations whose components at $X \in \mathbf{C}$ are the projection functions $F(X) \xleftarrow{\pi_1} F(X) \times G(X) \xrightarrow{\pi_2} G(X)$.  


Theorem. For each small category $\mathcal{C}$, the category $\text{Set}^{\mathcal{C}^{\text{op}}}$ of presheaves is cartesian closed.

Proof sketch.

We can work out what the value of the exponential $G^F \in \text{Set}^{\mathcal{C}^{\text{op}}}$ at $X \in \mathcal{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \cong \text{Set}^{\mathcal{C}^{\text{op}}}(yX, G^F) \cong \text{Set}^{\mathcal{C}^{\text{op}}}(yX \times F, G)$$

Yoneda Lemma

universal property of the exponential
**Theorem.** For each small category $\mathcal{C}$, the category $\text{Set}^{\mathcal{C}^{\text{op}}}$ of presheaves is cartesian closed.

**Proof sketch.**

We can work out what the value of the exponential $G^F \in \text{Set}^{\mathcal{C}^{\text{op}}}$ at $X \in \mathcal{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \cong \text{Set}^{\mathcal{C}^{\text{op}}}(yX, G^F) \cong \text{Set}^{\mathcal{C}^{\text{op}}}(yX \times F, G)$$

We take the set $\text{Set}^{\mathcal{C}^{\text{op}}}(yX \times F, G)$ to be the definition of the value of $G^F$ at $X$...
Exponential objects in $\text{Set}^{\text{C}^{\text{op}}}$:

$$G^F(X) \triangleq \text{Set}^{\text{C}^{\text{op}}}(yX \times F, G)$$

Given $Y \rightarrow X$ in $\text{C}$, we have $yY \rightarrow yX$ in $\text{Set}^{\text{C}^{\text{op}}}$ and hence

$$G^F(Y) \triangleq \text{Set}^{\text{C}^{\text{op}}}(yY \times F, G) \rightarrow \text{Set}^{\text{C}^{\text{op}}}(yX \times F, G) \triangleq G^F(X)$$

$$\theta \mapsto \theta \circ (yf \times \text{id}_F)$$

We define

$$G^F(f) \triangleq (yf \times \text{id}_F)$$

Have to check that these definitions make $G^F$ into a functor $\text{C}^{\text{op}} \rightarrow \text{Set}$. 
Application morphisms in $\text{Set}^{\text{Cop}}$:

Given $F, G \in \text{Set}^{\text{Cop}}$, the application morphism

$$\text{app} : G^F \times F \to G$$

is the natural transformation whose component at $X \in C$ is the function

$$(G^F \times F)(X) \triangleq G^F(X) \times F(X) \triangleq \text{Set}^{\text{Cop}}(yX \times F, G) \times F(X) \xrightarrow{\text{app}_X} G(X)$$

defined by

$$\text{app}_X(\theta, x) \triangleq \theta_X(\text{id}_X, x)$$

Have to check that this is natural in $X$. 
Currying operation in $\textbf{Set}^{\mathbf{C}^{\mathbf{op}}}$:

$$\left( H \times F \xrightarrow{\theta} G \right) \mapsto \left( H \xrightarrow{\text{cur } \theta} G^{F} \right)$$

Given $H \times F \xrightarrow{\theta} G$ in $\textbf{Set}^{\mathbf{C}^{\mathbf{op}}}$, the component of $\text{cur } \theta$ at $X \in \mathbf{C}$

$$H(X) \xrightarrow{(\text{cur } \theta)_X} G^{F}(X) \triangleq \textbf{Set}^{\mathbf{C}^{\mathbf{op}}}(yX \times F, G)$$

is the function mapping each $z \in H(X)$ to the natural transformation $yX \times F \rightarrow G$ whose component at $Y \in \mathbf{C}$ is the function

$$(yX \times F)(Y) \triangleq \mathbf{C}(Y, X) \times F(Y) \rightarrow G(Y)$$

defined by

$$(((\text{cur } \theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)$$
Currying operation in $\text{Set}^{\text{C}^{\text{op}}}$:

$$(H \times F \xrightarrow{\theta} G) \mapsto (H \xrightarrow{\text{cur } \theta} G^F)$$

Have to check that this is natural in $Y$,
then that $(\text{cur } \theta)_X$ is natural in $X$,
then that $\text{cur } \theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ in $\text{Set}^{\text{C}^{\text{op}}}$ satisfying $\text{app } \circ (\varphi \times \text{id}_F) = \theta$. 

$$(((\text{cur } \theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)$$
Theorem. For each small category $\mathcal{C}$, the category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ of presheaves is cartesian closed.

So we can interpret simply typed lambda calculus in any presheaf category.

More than that, presheaf categories (usefully) model dependently-typed languages.