STLC equations take the form $\Gamma \vdash s = t : A$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\Gamma \vdash s : A]$ and $M[\Gamma \vdash t : A]$ are equal $\mathsf{C}$-morphisms $M[\Gamma] \to M[A]$.

**Qu:** which equations are always satisfied in any ccc?

**Ans:** $\beta\eta$-equivalence…
The relation $\Gamma \vdash s =_{\beta \eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- **$\beta$-conversions**

\[
\begin{align*}
\Gamma, x : A & \vdash t : B \\
\Gamma & \vdash (\lambda x : A. t)s =_{\beta\eta} t[s/x] : B
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash s : A \\
\Gamma & \vdash t : B \\
\Gamma & \vdash \text{fst}(s, t) =_{\beta\eta} s : A
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash s : A \\
\Gamma & \vdash t : B \\
\Gamma & \vdash \text{snd}(s, t) =_{\beta\eta} t : B
\end{align*}
\]
The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- **$\beta$-conversions**
- **$\eta$-conversions**

\[
\begin{array}{ll}
\Gamma \vdash t : A \rightarrow B & x \text{ does not occur in } t \\
\hline
\Gamma \vdash t =_{\beta\eta} (\lambda x : A. t x) : A \rightarrow B
\end{array}
\]

\[
\begin{array}{ll}
\Gamma \vdash t : A \times B \\
\hline
\Gamma \vdash t =_{\beta\eta} (\text{fst } t, \text{snd } t) : A \times B
\end{array}
\]

\[
\begin{array}{ll}
\Gamma \vdash t : \text{unit} \\
\hline
\Gamma \vdash t =_{\beta\eta} () : \text{unit}
\end{array}
\]
The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions
- $\eta$-conversions
- congruence rules

\[
\begin{align*}
\Gamma, x : A \vdash t =_{\beta\eta} t' : B \\
\Gamma \vdash \lambda x : A. t =_{\beta\eta} \lambda x : A. t' : A \rightarrow B
\end{align*}
\]
\[
\begin{align*}
\Gamma \vdash s =_{\beta\eta} s' : A \rightarrow B \\
\Gamma \vdash t =_{\beta\eta} t' : A
\end{align*}
\]
\[
\Gamma \vdash s \; t =_{\beta\eta} s' t' : B
\]

etc
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions
- $\eta$-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

\[
\begin{align*}
\Gamma \vdash t : A & \quad \Gamma \vdash s =_{\beta\eta} t : A \\
\Gamma \vdash t =_{\beta\eta} t : A & \quad \Gamma \vdash t =_{\beta\eta} s : A \\
\Gamma \vdash r =_{\beta\eta} s : A & \quad \Gamma \vdash s =_{\beta\eta} t : A \\
\Gamma \vdash r =_{\beta\eta} t : A & \quad \Gamma \vdash r =_{\beta\eta} t : A
\end{align*}
\]
STLC $\beta\eta$-Equality

**Soundness Theorem** for semantics of STLC in a ccc. If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$$

are equal $C$-morphisms $M[\Gamma] \to M[A]$.

**Proof** is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of $\beta$-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\Gamma \vdash (\lambda x : A. t)s : B] = M[\Gamma \vdash t[s/x] : B]$$
Suppose

\[ M[\Gamma] = X \]
\[ M[A] = Y \]
\[ M[B] = Z \]
\[ M[\Gamma, x : A \vdash t : B] = f : X \times Y \to Z \]
\[ M[\Gamma \vdash s : A] = g : X \to Z \]

Then

\[ M[\Gamma \vdash \lambda x : A. t : A \to B] = \text{cur } f : X \to Z^Y \]

and hence

\[ M[\Gamma \vdash (\lambda x : A. t)s : B] \]
\[ = \text{app } \circ \langle \text{cur } f , g \rangle \]
\[ = \text{app } \circ (\text{cur } f \times \text{id}_Y) \circ \langle \text{id}_X , g \rangle \]  # since \((a \times b) \circ \langle c , d \rangle = \langle a \circ c , b \circ d \rangle\)
\[ = f \circ \langle \text{id}_X , g \rangle \]  # by definition of \text{cur } f
\[ = M[\Gamma \vdash t[s/x] : B] \]  # by the Substitution Theorem

as required.
The internal language of a ccc, $\mathcal{C}$

- one ground type for each $\mathcal{C}$-object $X$
- for each $X \in \mathcal{C}$, one constant $f^X$ for each $\mathcal{C}$-morphism $f : 1 \to X$ (“global element” of the object $X$)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of $\mathcal{C}$ using its cartesian closed structure, but in an “element-theoretic” way.

For example…
Example

In any ccc $C$, for any $X, Y, Z \in C$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$
Example

In any ccc \( C \), for any \( X, Y, Z \in C \) there is an isomorphism

\[ Z^{(X \times Y)} \cong (Z^Y)^X \]

which in the internal language of \( C \) is described by the terms

\[ \Diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z)) \]
\[ \Diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z) \]

where

\[
\begin{align*}
s & \triangleq \lambda f : (X \times Y) \to Z. \lambda x : X. \lambda y : Y. f(x, y) \\
t & \triangleq \lambda g : X \to (Y \to Z). \lambda z : X \times Y. g(fst z)(snd z)
\end{align*}
\]

and

which satisfy

\[
\begin{align*}
\Diamond, f : (X \times Y) \to Z & \vdash t(s f) =_{\beta \eta} f \\
\Diamond, g : X \to (Y \to Z) & \vdash s(t g) =_{\beta \eta} g
\end{align*}
\]
The **Soundness Theorem** has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc \( F \) (the free ccc for that language) with an interpretation function \( M \) so that \( \Gamma \vdash s =_{\beta\eta} t : A \) is provable iff \( M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A] \) in \( F \).
Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc (the free ccc for that language) with an interpretation function $M$ so that $\Gamma \vdash s \equiv_{\beta\eta} t : A$ is provable iff $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$ in $F$.

- **F-objects** are the STLC types over the given set of ground types
- **F-morphisms** $A \rightarrow B$ are equivalence classes of STLC terms $t$ satisfying $\Diamond \vdash t : A \rightarrow B$ (so $t$ is a closed term—it has no free variables) with respect to the equivalence relation equating $s$ and $t$ if $\Diamond \vdash s \equiv_{\beta\eta} t : A \rightarrow B$ is provable.
- identity morphism on $A$ is the equivalence class of $\Diamond \vdash \lambda x : A. x : A \rightarrow A$.
- composition of a morphism $A \rightarrow B$ represented by $\Diamond \vdash s : A \rightarrow B$ and a morphism $B \rightarrow C$ represented by $\Diamond \vdash t : B \rightarrow C$ is represented by $\Diamond \vdash \lambda x : A. t(s x) : A \rightarrow C$. 
Curry-Howard correspondence

<table>
<thead>
<tr>
<th>Logic</th>
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<tbody>
<tr>
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</tr>
<tr>
<td>proofs</td>
<td>↔ terms</td>
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</tbody>
</table>

E.g. IPL *versus* STLC.
Curry-Howard for IPL vs STLC

Proof of $\Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

where $\Phi = \Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$
Curry-Howard for IPL vs STLC

and a corresponding STLC term

where $\Phi = \Diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$
# Curry-Howard-Lawvere/Lambek correspondence

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E.g. IPL *versus* STLC *versus* CCCs
Curry-Howard-Lawvere/Lambek correspondence

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E.g. IPL versus STLC versus CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.