#### Lecture 9

### STLC equations

take the form  $\Gamma \vdash s = t : A$  where  $\Gamma \vdash s : A$  and  $\Gamma \vdash t : A$  are provable.

Such an equation is satisfied by the semantics in a ccc if  $M[\Gamma \vdash s : A]$  and  $M[\Gamma \vdash t : A]$  are equal C-morphisms  $M[\Gamma] \rightarrow M[A]$ .

Qu: which equations are always satisfied in any ccc? Ans:  $\beta\eta$ -equivalence...

The relation  $\Gamma \vdash s =_{\beta\eta} t : A$  (where  $\Gamma$  ranges over typing environments, *s* and *t* over terms and *A* over types) is inductively defined by the following rules:

#### $\beta$ -conversions

 $\begin{array}{|c|c|c|c|c|}\hline \Gamma, x : A \vdash t : B & \Gamma \vdash s : A \\\hline \Gamma \vdash (\lambda x : A, t)s =_{\beta\eta} t[s/x] : B \\\hline \hline \Gamma \vdash fst(s, t) =_{\beta\eta} s : A \\\hline \hline \Gamma \vdash s : A & \Gamma \vdash t : B \\\hline \Gamma \vdash snd(s, t) =_{\beta\eta} t : B \\\hline \end{array}$ 

• 
$$\beta$$
-conversions

$$\begin{array}{c} \Gamma \vdash t : A \Rightarrow B & x \text{ does not occur in } t \\ \hline \Gamma \vdash t =_{\beta\eta} (\lambda x : A. t x) : A \Rightarrow B \end{array} \\ \hline \\ \frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\text{fst } t, \text{ snd } t) : A \times B} & \begin{array}{c} \Gamma \vdash t : \text{ unit} \\ \hline \Gamma \vdash t =_{\beta\eta} () : \text{ unit} \end{array} \end{array}$$

- $\blacktriangleright$   $\beta$ -conversions
- $\blacktriangleright$   $\eta$ -conversions
- congruence rules

$$\Gamma, x : A \vdash t =_{\beta\eta} t' : B$$

$$\Gamma \vdash \lambda x : A. t =_{\beta\eta} \lambda x : A. t' : A \rightarrow B$$

$$\Gamma \vdash s =_{\beta\eta} s' : A \rightarrow B \qquad \Gamma \vdash t =_{\beta\eta} t' : A$$

$$\Gamma \vdash s t =_{\beta\eta} s' t' : B$$
etc

- $\beta$ -conversions
- $\blacktriangleright$   $\eta$ -conversions
- congruence rules

$$=_{\beta\eta} \text{ is reflexive, symmetric and transitive}$$

$$\boxed{\Gamma \vdash t : A} \qquad \boxed{\Gamma \vdash s =_{\beta\eta} t : A} \qquad \boxed{\Gamma \vdash s =_{\beta\eta} t : A} \qquad \boxed{\Gamma \vdash t =_{\beta\eta} s : A} \qquad \boxed{\Gamma \vdash r =_{\beta\eta} s : A} \qquad \boxed{\Gamma \vdash r =_{\beta\eta} t : A} \qquad \boxed{\Gamma \vdash r =$$

**Soundness Theorem** for semantics of STLC in a ccc. If  $\Gamma \vdash s =_{\beta\eta} t : A$  is provable, then in any ccc

 $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$ 

are equal C-morphisms  $M[[\Gamma]] \rightarrow M[[A]]$ .

**Proof** is by induction on the structure of the proof of  $\Gamma \vdash s =_{\beta\eta} t : A$ .

Here we just check the case of  $\beta$ -conversion for functions.

So suppose we have  $\Gamma, x : A \vdash t : B$  and  $\Gamma \vdash s : A$ . We have to see that

 $M[\![\Gamma \vdash (\lambda x : A. t)s : B]\!] = M[\![\Gamma \vdash t[s/x] : B]\!]$ 

Suppose 
$$M[[\Gamma]] = X$$
  
 $M[[A]] = Y$   
 $M[[B]] = Z$   
 $M[[\Gamma, x : A \vdash t : B]] = f : X \times Y \rightarrow Z$   
 $M[[\Gamma \vdash s : A]] = g : X \rightarrow Z$ 

Then

$$M[\![\Gamma \vdash \lambda x : A.t : A \twoheadrightarrow B]\!] = \operatorname{cur} f : X \longrightarrow Z^Y$$

and hence

$$M[\Gamma \vdash (\lambda x : A. t)s : B]]$$
  
= app \cur f, g  
= app \cur f \times id\_Y) \circ \langle id\_X, g  
= f \circ \langle id\_X, g  
= M[[\Gamma \dot t[s/x]] : B]]

as required.

since  $(a \times b) \circ \langle c, d \rangle = \langle a \circ c, b \circ d \rangle$ by definition of cur fby the <u>Substitution Theorem</u> The internal language of a ccc, C
one ground type for each C-object X
for each X ∈ C, one constant f<sup>X</sup> for each C-morphism f : 1 → X ("global element" of the object X)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of C using its cartesian closed structure, but in an "element-theoretic" way.

For example...

### Example

#### In any ccc C, for any $X, Y, Z \in C$ there is an isomorphism

 $Z^{(X \times Y)} \cong (Z^Y)^X$ 

## Example

In any ccc **C**, for any  $X, Y, Z \in \mathbf{C}$  there is an isomorphism  $Z^{(X \times Y)} \cong (Z^Y)^X$ 

which in the internal language of **C** is described by the terms

 $\diamond \vdash s : ((X \times Y) \Rightarrow Z) \Rightarrow (X \Rightarrow (Y \Rightarrow Z))$  $\diamond \vdash t : (X \Rightarrow (Y \Rightarrow Z)) \Rightarrow ((X \times Y) \Rightarrow Z)$ 

where 
$$\begin{cases} s &\triangleq \lambda f : (X \ge Y) \Rightarrow Z. \ \lambda x : X. \ \lambda y : Y. \ f(x, y) \\ t &\triangleq \lambda g : X \Rightarrow (Y \Rightarrow Z). \ \lambda z : X \ge Y. \ g \ (\texttt{fst} z) \ (\texttt{snd} z) \end{cases} \text{ and} \\ \text{which satisfy} \begin{cases} \diamond, f : (X \ge Y) \Rightarrow Z \vdash t(s \ f) =_{\beta\eta} f \\ \diamond, g : X \Rightarrow (Y \Rightarrow Z) \vdash s(t \ g) =_{\beta\eta} g \end{cases}$$

#### Free cartesian closed categories

The Soundness Theorem has a converse-completeness.

In fact for a given set of ground types and typed constants there is a single ccc **F** (the free ccc for that language) with an interpretation function *M* so that  $\Gamma \vdash s =_{\beta\eta} t : A$  is provable iff  $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$  in **F**.

#### Free cartesian closed categories

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- F-objects are the STLC types over the given set of ground types
- F-morphisms  $A \rightarrow B$  are equivalence classes of STLC terms t satisfying  $\diamond \vdash t : A \rightarrow B$  (so t is a *closed* term—it has no free variables) with respect to the equivalence relation equating s and t if  $\diamond \vdash s =_{\beta\eta} t : A \rightarrow B$  is provable.
- identity morphism on A is the equivalence class of  $\diamond \vdash \lambda x : A \cdot x : A \rightarrow A$ .
- composition of a morphism  $A \to B$  represented by  $\diamond \vdash s : A \to B$  and a morphism  $B \to C$  represented by  $\diamond \vdash t : B \to C$  is represented by  $\diamond \vdash \lambda x : A. t(s x) : A \to C$ .

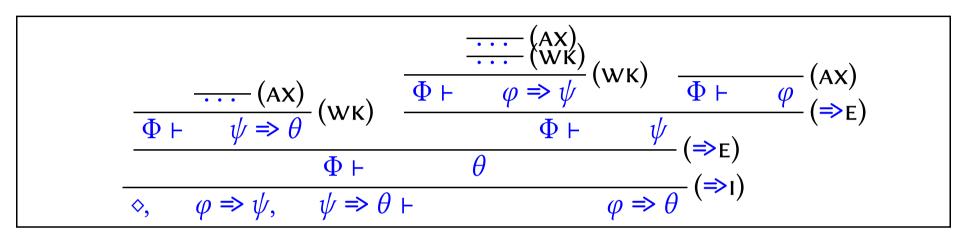
## Curry-Howard correspondence

	Туре		
Logic		Theory	
propositions	$\leftrightarrow$	types	
proofs	$\leftrightarrow$	terms	

E.g. IPL versus STLC.

### Curry-Howard for IPL vs STLC

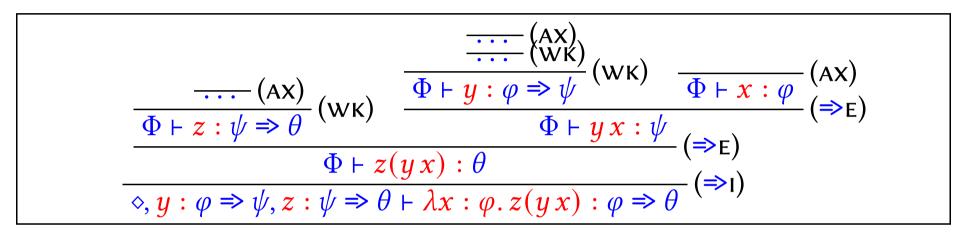
Proof of  $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$  in IPL



where  $\Phi = \diamond$ ,  $\varphi \Rightarrow \psi$ ,  $\psi \Rightarrow \theta$ ,  $\varphi$ 

## Curry-Howard for IPL vs STLC

#### and a corresponding STLC term



where  $\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$ 

## Curry-Howard-Lawvere/Lambek correspondence

Logic	Type Theory		Category Theory	
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL versus STLC versus CCCs

# Curry-Howard-Lawvere/Lambek correspondence

		Туре		Category
Logic		Theory		Theory
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

#### E.g. IPL versus STLC versus CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.