Lecture 2
Recall

A category $C$ is specified by

- a set $\text{obj } C$ whose elements are called $C$-objects
- for each $X, Y \in \text{obj } C$, a set $C(X, Y)$ whose elements are called $C$-morphisms from $X$ to $Y$
- a function assigning to each $X \in \text{obj } C$ an element $\text{id}_X \in C(X, X)$ called the identity morphism for the $C$-object $X$
- a function assigning to each $f \in C(X, Y)$ and $g \in C(Y, Z)$ (where $X, Y, Z \in \text{obj } C$) an element $g \circ f \in C(X, Z)$ called the composition of $C$-morphisms $f$ and $g$ and satisfying associativity and unity properties.
Example: category of pre-orders, Preord

- objects are sets $P$ equipped with a pre-order $\sqsubseteq$, i.e. a binary relation on $P$ that is
  - reflexive: $\forall x \in P, \ x \sqsubseteq x$
  - transitive: $\forall x, y, z \in P, \ x \sqsubseteq y \land y \sqsubseteq z \Rightarrow x \sqsubseteq z$

A partial order is a pre-order that is also
- anti-symmetric: $\forall x, y \in P, \ x \sqsubseteq y \land y \sqsubseteq x \Rightarrow x = y$
Example: category of pre-orders, \textbf{Preord}

- objects are sets $P$ equipped with a pre-order $\sqsubseteq$
- morphisms: $\text{Preord}(((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \}$

\[ \forall x, x' \in P_1, x \sqsubseteq_1 x' \Rightarrow f x \sqsubseteq_2 f x' \]
Example: category of pre-orders, $\text{Preord}$

- objects are sets $P$ equipped with a pre-order $\preceq$
- morphisms: $\text{Preord}((P_1, \preceq_1), (P_2, \preceq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \}$
- identities and composition: as for $\text{Set}$

Q: why is this well-defined?
A: because the set of monotone functions contains identity functions and is closed under composition.
Example: category of pre-orders, $\text{Preord}$

- objects are sets $P$ equipped with a pre-order $\sqsubseteq$
- morphisms: $\text{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \}$
- identities and composition: as for $\text{Set}$

Pre- and partial orders are relevant to the denotational semantics of programming languages (among other things).
Example: category of monoids, $\textbf{Mon}$

- objects are monoids $(M, \cdot, e)$ — set $M$ equipped with a binary operation $\_ \cdot \_ : M \times M \to M$ which is associative $\forall x, y, z \in M, \ x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- has $e$ as its unit $\forall x \in M, \ e \cdot x = x = x \cdot e$

CS-relevant example of a monoid: $(\text{List } \Sigma, @, \text{nil})$ where

\[
\begin{align*}
\text{List } \Sigma &= \text{set of finite lists of elements of set } \Sigma \\
@ &= \text{list concatenation} \\
\text{nil} @ \ell &= \ell \\
(a :: \ell) @ \ell' &= a :: (\ell @ \ell') \\
\text{nil} &= \text{empty list}
\end{align*}
\]
Example: category of monoids, $\textbf{Mon}$

- **objects** are monoids $(M, \cdot, e)$
- **morphisms**: $\textbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \textbf{Set}(M_1, M_2) \mid f \, e_1 = e_2 \wedge \forall x, y \in M_1, f(x \cdot_1 y) = (f \, x) \cdot_2 (f \, y) \}$

It’s common to denote a monoid $(M, \cdot, e)$ just by its underlying set $M$, leaving $\cdot$ and $e$ implicit (hence the same notation gets used for different instances of monoid operations).
Example: category of monoids, \( \text{Mon} \)

- objects are monoids \((M, \cdot, e)\)
- morphisms: \( \text{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \text{Set}(M_1, M_2) \mid f \, e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (f \, x) \cdot_2 (f \, y) \}\)
- identities and composition: as for \( \text{Set} \)

Q: why is this well-defined?
A: because the set of functions that are monoid morphisms contains identity functions and is closed under composition.
Example: category of monoids, $\textbf{Mon}$

- objects are monoids $(M, \cdot, e)$
- morphisms: $\textbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \text{Set}(M_1, M_2) \mid f e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y) \}$
- identities and composition: as for $\text{Set}$

Monoids are relevant to automata theory (among other things).
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(C_P\) by taking

- objects \(\text{obj } C_P = P\)
- morphisms \(C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)

(where 1 is some fixed one-element set and \(\emptyset\) is the empty set)
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(C_P\) by taking

- objects \(\text{obj } C_P = P\)
- morphisms \(C_P(x, y) \triangleq \begin{cases} 
1 & \text{if } x \sqsubseteq y \\
\emptyset & \text{if } x \not\sqsubseteq y 
\end{cases}\)

- identity morphisms and composition are uniquely determined (why?)
Example: each pre-order determines a category

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- identity morphisms and composition are uniquely determined (why?)

E.g. when \((P, \sqsubseteq)\) has just one element 0

\[ C_P = \begin{array}{ccc}
0 & \xrightarrow{\text{id}_0} & 0 \\
\end{array}\]

one object, one morphism
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\),
we get a category \(C_P\) by taking

- **objects** \(\text{obj } C_P = P\)
- **morphisms** \(C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)
- **identity morphisms** and composition are uniquely determined *(why?)*

E.g. when \((P, \sqsubseteq)\) has just two elements \(0 \sqsubseteq 1\)

\[
C_P = \begin{array}{c}
\text{id}_0 \quad 0 \quad \rightarrow \quad 1 \quad \text{id}_1 \\
\text{two objects, one non-identity morphism}
\end{array}
\]
Example: each pre-order determines a category

Given a pre-ordered set $(P, \sqsubseteq)$, we get a category $C_P$ by taking

- objects $\text{obj } C_P = P$
- morphisms $C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$
- identity morphisms and composition are uniquely determined (why?)

Example of a finite category that does not arise from a pre-ordered set:

\[
\begin{array}{c}
id_0 & 0 & 1 & \text{id}_1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 1 & 1 & 0
\end{array}
\]

two objects, two non-identity morphisms
Example: each monoid determines a category

Given a monoid \((M, \cdot, e)\), we get a category \(C_M\) by taking

- objects: \(\text{obj } C_M = 1 = \{0\}\) (one-element set)
- morphisms: \(C_M(0, 0) = M\)
- identity morphism: \(\text{id}_0 = e \in M = C_M(0, 0)\)
- composition of \(f \in C_M(0, 0)\) and \(g \in C_M(0, 0)\) is \(g \cdot f \in M = C_M(0, 0)\)
Definition of isomorphism

Let $C$ be a category. A $C$-morphism $f : X \rightarrow Y$ is an isomorphism if there is some $g : Y \rightarrow X$ for which

$$
\begin{align*}
X & \xrightarrow{f} Y \\
\downarrow{g} & \quad \downarrow{id_Y} \\
X & \xrightarrow{id_X} X
\end{align*}
$$

is a commutative diagram.
Definition of isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

- Such a $g$ is uniquely determined by $f$ (why?) and we write $f^{-1}$ for it.
- Given $X, Y \in \mathbf{C}$, if such an $f$ exists, we say the objects $X$ and $Y$ are isomorphic in $\mathbf{C}$ and write $X \cong Y$.

(There may be many different $f$ that witness the fact that $X$ and $Y$ are isomorphic.)
Theorem. A function $f \in \text{Set}(X, Y)$ is an isomorphism in the category \textbf{Set} iff $f$ is a bijection, that is

- **injective:** $\forall x, x' \in X, \ f x = f x' \Rightarrow x = x'$
- **surjective:** $\forall y \in Y, \exists x \in X, \ f x = y$

Proof...
**Theorem.** A function \( f \in \text{Set}(X, Y) \) is an isomorphism in the category \textbf{Set} iff \( f \) is a bijection, that is

- **injective:** \( \forall x, x' \in X, \ f x = f x' \Rightarrow x = x' \)
- **surjective:** \( \forall y \in Y, \exists x \in X, \ f x = y \)

Proof...

**Theorem.** A monoid morphism \( f \in \text{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \) is an isomorphism in the category \textbf{Mon} iff \( f \in \text{Set}(M_1, M_2) \) is a bijection.

Proof...
Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.
Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

**Theorem.** A monotone function \( f \in \text{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \) is an isomorphism in the category **Poset** iff \( f \in \text{Set}(P_1, P_2) \) is a surjective function satisfying

- **reflective:** \( \forall x, x' \in P_1, \ f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x' \)

**Proof...**
Define $\textbf{Poset}$ to be the category whose objects are posets = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category $\textbf{Preord}$ of pre-ordered sets.

**Theorem.** A monotone function $f \in \textbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category $\textbf{Poset}$ iff $f \in \textbf{Set}(P_1, P_2)$ is a surjective function satisfying

- ▶ reflective: $\forall x, x' \in P_1, \; f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'$

**Proof...**

(Why does this characterisation not work for $\textbf{Preord}$?)