

Category Theory

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Part II Unit of Assessment
Part III and MPhil. ACS Module L108

Course web page

Go to

<https://www.cl.cam.ac.uk/teaching/2021/CatTheory/>

<https://www.cl.cam.ac.uk/teaching/2021/L108/>

for

- ▶ these slides and lecture recordings
- ▶ exercise sheets and details of examples classes
(trying the exercises is essential!)
- ▶ pointers to some additional material

Recommended text for the course is:

[Awodey] Steve Awodey, *Category theory*,
Oxford University Press (2nd ed.), 2010.

Assessment

- ▶ **A graded exercise sheet** (25% of the final mark).
issued in lecture 10 with a one week deadline
- ▶ **A take-home test** (75% of the final mark).
issued after the end of the course

See course web page for dates and deadlines.

Lecture 1

What is category theory?

What we are probably seeking is a “purer” view of **functions**: a theory of functions in themselves, not a theory of functions derived from sets. What, then, is a pure theory of functions? Answer: **category theory**.

Dana Scott, *Relating theories of the λ -calculus*, p406

set theory gives an “element-oriented” account of mathematical structure, whereas

category theory takes a ‘function-oriented’ view – understand structures not via their elements, but by how they transform, i.e. via **morphisms**.

(Both theories are part of Logic, broadly construed.)

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

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Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its “dual”

Presented to the Society, September 8, 1942; received by the editors May 15, 1945.

Category Theory emerges

1945 Eilenberg[†] and MacLane[†]
General Theory of Natural Equivalences,
Trans AMS 58, 231–294

(algebraic topology, abstract algebra)

1950s Grothendieck[†] (algebraic geometry)

1960s Lawvere (logic and foundations)

1970s Joyal and Tierney[†] (elementary topos theory)

1980s Dana Scott, Plotkin

(semantics of programming languages)

Lambek[†] (linguistics)

Category Theory and Computer Science

“Category theory has...become part of the standard “tool-box” in many areas of theoretical informatics, from programming languages to automata, from process calculi to Type Theory.”

Dagstuhl Perspectives Workshop on *Categorical Methods at the Crossroads*
April 2014

This course

basic concepts of category theory

adjunction ← **natural transformation**

category → **functor**

applied to {
 propositional logic
 typed lambda-calculus
 functional programming

Definition

A **category** \mathbf{C} is specified by

- ▶ a set $\text{obj } \mathbf{C}$ whose elements are called **\mathbf{C} -objects**
- ▶ for each $X, Y \in \text{obj } \mathbf{C}$, a set $\mathbf{C}(X, Y)$ whose elements are called **\mathbf{C} -morphisms from X to Y**

(so far, that is just what some people call a **directed graph**)

Definition

A **category** \mathbf{C} is specified by

- ▶ a set $\text{obj } \mathbf{C}$ whose elements are called **C-objects**
- ▶ for each $X, Y \in \text{obj } \mathbf{C}$, a set $\mathbf{C}(X, Y)$ whose elements are called **C-morphisms from X to Y**
- ▶ a function assigning to each $X \in \text{obj } \mathbf{C}$ an element $\text{id}_X \in \mathbf{C}(X, X)$ called the **identity morphism** for the **C-object** X
- ▶ a function assigning to each $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ (where $X, Y, Z \in \text{obj } \mathbf{C}$) an element $g \circ f \in \mathbf{C}(X, Z)$ called the **composition** of **C-morphisms** f and g and satisfying...

Definition, continued

satisfying...

- ▶ **associativity**: for all $X, Y, Z, W \in \text{obj } \mathbf{C}$,
 $f \in \mathbf{C}(X, Y)$, $g \in \mathbf{C}(Y, Z)$ and $h \in \mathbf{C}(Z, W)$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- ▶ **unity**: for all $X, Y \in \text{obj } \mathbf{C}$ and $f \in \mathbf{C}(X, Y)$

$$\text{id}_Y \circ f = f = f \circ \text{id}_X$$

Example: category of sets, **Set**

- ▶ **obj Set** = some fixed universe of sets

(more on universes later)

- ▶ **Set**(X, Y) =

$\{f \subseteq X \times Y \mid f \text{ is single-valued and total}\}$

Cartesian product of sets X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Equality of ordered pairs:

$$(x, y) = (x', y') \Leftrightarrow x = x' \wedge y = y'$$

Example: category of sets, **Set**

- ▶ **obj Set** = some fixed universe of sets

(more on universes later)

- ▶ **Set**(X, Y) =

$\{f \subseteq X \times Y \mid f \text{ is single-valued and total}\}$

$$\forall x \in X, \forall y, y' \in Y, \\ (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'$$

$$\forall x \in X, \exists y \in Y, \\ (x, y) \in f$$

Example: category of sets, **Set**

- ▶ **obj Set** = some fixed universe of sets

(more on universes later)

- ▶ **Set**(X, Y) =

$\{f \subseteq X \times Y \mid f \text{ is single-valued and total}\}$

- ▶ $\text{id}_X = \{(x, x) \mid x \in X\}$

- ▶ composition of $f \in \text{Set}(X, Y)$ and $g \in \text{Set}(Y, Z)$ is

$$g \circ f = \{(x, z) \mid \exists y \in Y, (x, y) \in f \wedge (y, z) \in g\}$$

(check that associativity and unity properties hold)

Example: category of sets, **Set**

Notation. Given $f \in \mathbf{Set}(X, Y)$ and $x \in X$, it is usual to write $f x$ (or $f(x)$) for the unique $y \in Y$ with $(x, y) \in f$.

Thus

$$\text{id}_X x = x$$

$$(g \circ f) x = g(f x)$$

Domain and codomain

Given a category \mathbf{C} ,

write $f : X \rightarrow Y$ or $X \xrightarrow{f} Y$

to mean that $f \in \mathbf{C}(X, Y)$,

in which case one says

object X is the **domain** of the morphism f

object Y is the **codomain** of the morphism f

and writes

$$X = \text{dom } f \quad Y = \text{cod } f$$

(Which category \mathbf{C} we are referring to is left implicit with this notation.)

Commutative diagrams

in a category \mathbf{C} :

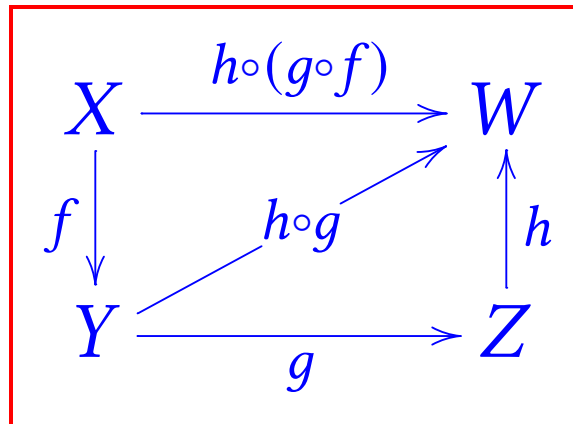
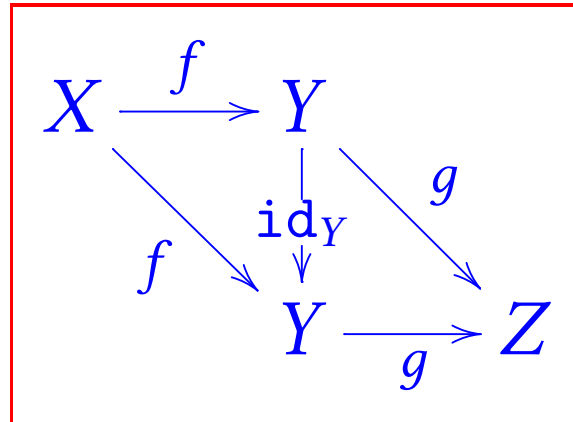
a **diagram** is

a directed graph whose vertices are \mathbf{C} -objects
and whose edges are \mathbf{C} -morphisms

and the diagram is **commutative** (or **commutes**) if
any two finite paths in the graph between any
two vertices determine equal morphisms in the
category under composition

Commutative diagrams

Examples:



Alternative notations

I will often just write

C for $\text{obj } C$

id for id_X

Some people write

$\text{Hom}_C(X, Y)$ for $C(X, Y)$

1_X for id_X

gf for $g \circ f$

I use “applicative order” for morphism composition;
other people use “diagrammatic order” and write

$f;g$ (or fg) for $g \circ f$

Alternative definition of category

The definition given here is “dependent-type friendly”.

See [Awodey, Definition 1.1] for an equivalent formulation:

One gives the whole set of morphisms $\text{mor } \mathbf{C}$
(in bijection with $\sum_{X, Y \in \text{obj } \mathbf{C}} \mathbf{C}(X, Y)$ in my definition)
plus functions

$$\text{dom}, \text{cod} : \text{mor } \mathbf{C} \rightarrow \text{obj } \mathbf{C}$$

$$\text{id} : \text{obj } \mathbf{C} \rightarrow \text{mor } \mathbf{C}$$

and a *partial* function for composition

$$_ \circ _ : \text{mor } \mathbf{C} \times \text{mor } \mathbf{C} \rightarrow \text{mor } \mathbf{C}$$

defined at (f, g) iff $\text{cod } f = \text{dom } g$

and satisfying the associativity and unity equations.