1. Recall (from Lecture 2) that a pre-ordered set \((P, \leq_P)\) determines a category \(C_P\) whose objects are the elements of \(P\) and whose morphism sets \(C_P(x, x')\) contain at most one element and do so iff \(x \leq_P x'\). Note that given two pre-ordered sets \((P, \leq_P)\) and \((Q, \leq_Q)\), a functor \(F : C_P \to C_Q\) is the same thing as a monotone function from \((P, \leq_P)\) to \((Q, \leq_Q)\).

(a) Given two such functors \(F, G : C_P \to C_Q\), how many natural transformations are there from \(F\) to \(G\)?

(b) Given monotone functions \(F : C_P \to C_Q\) and \(G : C_Q \to C_P\), give a property of \(F\) and \(G\) which ensures that, regarding them as functors, \(G\) is right adjoint to \(F\).

2. Recall that \(\text{Preord}\) denotes the category of pre-ordered sets and monotone functions. For each set \(X\), let \((\text{Pow} X, \subseteq) \in \text{obj Preord}\) be the set of all subsets of \(X\) partially ordered by inclusion.

Given a function \(f : X \to Y\), let \(f^{-1} : \text{Pow} Y \to \text{Pow} X\) be the function that maps each subset \(B \subseteq Y\) to the subset \(f^{-1}B \triangleq \{x \in X \mid f(x) \in B\} \subseteq X\).

(a) Show that \(f^{-1}\) is a monotone function and hence gives a morphism \((\text{Pow} Y, \subseteq) \to (\text{Pow} X, \subseteq)\) in \(\text{Preord}\).

(b) Regarding \(f^{-1}\) as a functor as in question (1), show that it has both left and right adjoints, given on objects by the following 'generalized quantifiers'

\[
\exists_f A \triangleq \{y \in Y \mid \exists x \in X, f(x) = y \land x \in A\}
\]

\[
\forall_f A \triangleq \{y \in Y \mid \forall x \in X, f(x) = y \Rightarrow x \in A\}
\]

(for all \(A \in \text{Pow} X\)). [Hint: use your answer to question 1b.]

3. A category \(C\) has pullbacks if for every pair of \(C\)-morphisms with a common codomain, \(Y \xrightarrow{f} X \leftarrow Z\), there is an object \(Y \times_Z Z\) and morphisms \(p, q\) making the following diagram commute in \(C\) (that is, \(f \circ p = g \circ q\))

\[
\begin{array}{ccc}
Y \times_Z Z & \xrightarrow{q} & Z \\
\downarrow p & & \downarrow g \\
Y & \xrightarrow{f} & X
\end{array}
\]

and with the following universal property:

For all \(Y \leftarrow W \xrightarrow{k} Z\) in \(C\) with \(f \circ h = g \circ k\), there is a unique morphism \(\ell \in C(W, Y \times_Z Z)\)
satisfying $p \circ \ell = h$ and $q \circ \ell = k$

(a) Show that $C$ has pullbacks iff for all $X \in \text{obj } C$ the slice category $C/X$ has binary products (see Lecture 14, slide 166).

(b) Show that if $C$ has a terminal object and pullbacks, then it has binary products.

(c) Suppose $C$ has pullbacks. Given $f \in C(Y, X)$, show that the mapping

$$f^*: \begin{array}{ccc} Z & \to & Y \times_X Y \times_X Z \\ g & \downarrow & \downarrow p \\ X & \to & Y \end{array}$$

is the object part of a functor $f^*: C/X \to C/Y$ between slice categories.

(d) Show that the functor $f^*$ in part (c) always has a left adjoint $f!: C/Y \to C/X$, which on objects sends $(W, h) \in \text{obj } (C/Y)$ to $f!(W, h) \triangleq (W, f \circ h) \in \text{obj } (C/X)$.

4. Suppose $(T, \eta, \mu)$ is a monad on a category $C$ (see Lecture 16). Thus $T : C \to C$ is a functor and $\eta : \text{id}_C \to T$ and $\mu : T \circ T \to T$ are natural transformations satisfying $\mu \circ T \eta = \text{id}_T = \mu \circ \eta_T$ and $\mu \circ \mu_T = \mu \circ T \mu$ (see Exercise Sheet 5, question 5 for the notation being used in those equations). The Kleisli category $C_T$ of the monad has the same objects as $C$; we will write $FX$ for the object of $C_T$ corresponding to an object $X \in \text{obj } C$. Given $X, Y \in \text{obj } C$, $C_T(FX, FY)$ is defined to be $C(X, FY)$.

(a) Complete the definition of $C_T$ by giving the definition of identity morphisms and composition satisfying the usual associativity and unity properties.

(b) Show that the mapping $X \in \text{obj } C \mapsto FX \in \text{obj } C_T$ extends to a functor $F : C \to C_T$.

(c) Show that the functor $F$ has a right adjoint $G : C_T \to C$.

(d) Show that the monad associated with the adjunction $F \dashv G$ (see Lecture 16) is $(T, \eta, \mu)$. 