Mathematical Methods for Computer Science

UNIVERSITY OF CAMBRIDGE

Computer Laboratory

Computer Science Tripos, Part IB

Michaelmas Term 2005

R. J. Gibbens

Problem sheet
Part (a): Limits and inequalities

William Gates Building
15 JJ Thomson Avenue
Cambridge
CB3 0FD

http://www.cl.cam.ac.uk/

Limits and inequalities

1. Suppose that X is a random variable with the $U(-1,1)$ distribution. Find the exact value of $\mathbb{P}(|X| > a)$ for each $a > 0$ and compare it to the upper bounds obtained from the Markov and Chebychev inequalities.

2. Let X be the random variable giving the number of heads obtained in a sequence of n fair coin flips. Compare the upper bounds on $\mathbb{P}(X > 3n/4)$ obtained from the Markov and Chebychev inequalities.

3. Let A_i $(i = 1, 2, \dots, n)$ be a collection of random events and set $N = \sum_{i=1}^{n} \mathbb{I}(A_i)$. By considering Markov's inequality applied to $\mathbb{P}(N \geq 1)$ show Boole's inequality, namely,

$$\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)\,.$$

4. Let $h : \mathbb{R} \to [0, \infty)$ be a non-negative function. Show that

$$\mathbb{P}(h(X) \geq a) \leq \frac{\mathbb{E}(h(X))}{a} \qquad \text{for all} \qquad a > 0\,.$$

   By making suitable choices of $h(x)$, show that we may obtain the Markov and Chebychev inequalities as special cases.

5. Show the following properties of the moment generating function.
   (a) If X has mgf $M_X(t)$ then $Y = aX + b$ has mgf $M_Y(t) = e^{bt} M_X(at)$.
   (b) If X and Y are independent then $X + Y$ has mgf $M_{X+Y}(t) = M_X(t) M_Y(t)$.
   (c) $\mathbb{E}(X^n) = M_X^{(n)}(0)$ where $M_X^{(n)}$ is the n^{th} derivative of M_X.
   (d) If X is a discrete random variable taking values $0, 1, 2, \dots$ with probability generating function $g_X(z) = \mathbb{E}(z^X)$ then $M_X(t) = g_X(e^t)$.

6. Let X be a random variable with moment generating function $M_X(t)$ which you may assume exists for any value of t. Show that for any $a > 0$

$$\mathbb{P}(X \leq a) \leq e^{-ta} M_X(t) \qquad \text{for all} \qquad t < 0\,.$$

7. Show that, if $X_n \xrightarrow{D} X$, where X is a degenerate random variable (that is, $\mathbb{P}(X = \mu) = 1$ for some constant μ) then $X_n \xrightarrow{P} X$.

8. Suppose that you estimate your monthly phone bill by rounding all amounts to the nearest pound. If all rounding errors are independent and distributed as $U(-0.5, 0.5)$, estimate the probability that the total error exceeds one pound when your bill has 12 items. How does this procedure suggest an approximate method for constructing Normal random variables?