7 — DIFFERENCE EQUATIONS

Many problems in Probability give rise to *difference equations*. Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics.

Anyone who has made a study of differential equations will know that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with.

Aside from Probability, Computer Scientists take an interest in difference equations for a number of reasons. For example, difference equations frequently arise when determining the cost of an algorithm in big-O notation. Since difference equations are readily handled by program, a standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation.

Classification of Difference Equations

As with differential equations, one can refer to the *order* of a difference equation and note whether it is *linear* or *non-linear* and whether it is *homogeneous* or *inhomogeneous*.

The present discussion will almost exclusively be confined to linear second order difference equations both homogeneous and inhomogeneous.

Notation Convention

A trivial example stems from considering the sequence of odd numbers starting from 1. The associated difference equation might be specified as:

$$f(n) = f(n-1) + 2 \qquad \text{given that} \quad f(1) = 1$$

In words: term n in the sequence is two more than term $n-1$. The proviso, $f(1) = 1$, constitutes an *initial condition*. The first term in the sequence is 1.

A term like $f(n)$ so strongly suggests a continuous function that many writers prefer to use a subscript notation. The present difference equation would be presented as:

$$u_n = u_{n-1} + 2 \qquad \text{given that} \quad u_1 = 1 \tag{7.1}$$

This is the notation which will be used below. It is strongly implicit that n is an integer.

In simple cases, a difference equation gives rise to an associated *auxiliary equation* (first explained in (7.2) overleaf). In the case of (7.1) the associated auxiliary equation is:

$$w^1 - 1 = 0$$

The highest power of the polynomial in w is 1 and, accordingly, (7.1) is said to be a first order difference equation. If the constant term 2 were absent from (7.1), the equation would be homogeneous but its presence makes it inhomogeneous.

Some standard techniques for solving elementary difference equations analytically will now be presented…

Second Order Homogeneous Linear Difference Equation — I

To solve:

$$u_n = u_{n-1} + u_{n-2} \qquad \text{given that} \quad u_0 = 1 \quad \text{and} \quad u_1 = 1$$

transfer all the terms to the left-hand side:

$$u_n - u_{n-1} - u_{n-2} = 0$$

The zero on the right-hand side signifies that this is a homogeneous difference equation.

Guess:

$$u_n = Aw^n$$

so:

$$Aw^n - Aw^{n-1} - Aw^{n-2} = 0$$

and:

$$w^2 - w - 1 = 0 \tag{7.2}$$

This is the auxiliary equation associated with the difference equation. Being a quadratic, the auxiliary equation signifies that the difference equation is of second order.

The two roots are readily determined:

$$w_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad w_2 = \frac{1-\sqrt{5}}{2}$$

For any A_1 substituting $A_1 w_1^n$ for u_n in $u_n - u_{n-1} - u_{n-2}$ yields zero.

For any A_2 substituting $A_2 w_2^n$ for u_n in $u_n - u_{n-1} - u_{n-2}$ yields zero.

This suggests a general solution:

$$u_n = A_1 w_1^n + A_2 w_2^n$$

Check by substituting into $u_n - u_{n-1} - u_{n-2}$ thus:

$$(A_1 w_1^n + A_2 w_2^n) - (A_1 w_1^{n-1} + A_2 w_2^{n-1}) - (A_1 w_1^{n-2} + A_2 w_2^{n-2})$$

This, rearranged, is:

$$A_1 w_1^{n-2}(w_1^2 - w_1 - 1) + A_2 w_2^{n-2}(w_2^2 - w_2 - 1)$$

and this is zero because both expressions in brackets are zero.

On substituting the values of w_1 and w_2 the general solution is:

$$u_n = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

but, by noting initial conditions, values for A_1 and A_2 can be obtained...
Note:

$$u_0 = 1 \quad \text{so} \quad A_1 + A_2 = 1 \quad \text{and} \quad A_2 = 1 - A_1$$

Likewise:

$$u_1 = 1 \quad \text{so} \quad \frac{A_1\big(1+\sqrt{5}\big) + (1-A_1)\big(1-\sqrt{5}\big)}{2} = 1$$

so:

$$A_1\big(1+\sqrt{5}\big) + \big(1-\sqrt{5}\big) - A_1\big(1-\sqrt{5}\big) = 2$$

$$A_1\big(1+\sqrt{5}-1+\sqrt{5}\big) = 2 - 1 + \sqrt{5}$$

$$A_1\big(2\sqrt{5}\big) = 1 + \sqrt{5}$$

Hence:

$$A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

and:

$$A_2 = 1 - A_1 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}-1-\sqrt{5}}{2\sqrt{5}} = \frac{-1+\sqrt{5}}{2\sqrt{5}} = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

In consequence:

$$u_n = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

giving:

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \tag{7.3}$$

as the final solution.

Observe that despite the $\sqrt{5}$s:

$$u_0 = 1, \quad u_1 = 1, \quad u_2 = 2, \quad u_3 = 3, \quad u_4 = 5, \quad \text{etc.}$$

Second Order Homogeneous Linear Difference Equation — II

To solve:

$$u_n = p\,u_{n+1} + q\,u_{n-1} \qquad \text{given that} \quad u_0 = 0, \quad u_l = 1 \quad \text{and} \quad p + q = 1$$

Transfer all the terms to the left-hand side:

$$p\,u_{n+1} - u_n + q\,u_{n-1} = 0$$

Guess:

$$u_n = Aw^n$$

so:

$$pAw^{n+1} - Aw^n + qAw^{n-1} = 0$$

$$pw^2 - w + q = 0$$

$$pw^2 - (p+q)w + q = 0$$

$$(w-1)(pw-q) = 0$$

The two roots are:

$$w_1 = 1 \qquad \text{and} \qquad w_2 = \frac{q}{p}$$

This suggests a general solution:

$$u_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n \qquad \text{provided} \quad p \neq q \tag{7.4}$$

Check by substituting into $p\,u_{n+1} - u_n + q\,u_{n-1}$ thus:

$$\left[pA_1(1)^{n+1} + pA_2\left(\frac{q}{p}\right)^{n+1}\right] - \left[A_1(1)^n + A_2\left(\frac{q}{p}\right)^n\right] + \left[qA_1(1)^{n-1} + qA_2\left(\frac{q}{p}\right)^{n-1}\right]$$

This, rearranged, is:

$$A_1[p - 1 + q] + A_2\left(\frac{q}{p}\right)^{n-1}\left[p\left(\frac{q}{p}\right)^2 - \frac{q}{p} + q\right]$$

which, given that $p + q = 1$, is:

$$A_2\left(\frac{q}{p}\right)^{n-1}\left[\frac{q^2}{p} - \frac{q}{p} + q\right] = A_2\left(\frac{q}{p}\right)^{n-1}\left[\frac{q}{p}(q-1) + q\right] = A_2\left(\frac{q}{p}\right)^{n-1}\left[\frac{q}{p}(-p) + q\right] = 0$$

The next step is to determine values for A_1 and A_2 in the general solution…

The general solution was determined to be:

$$u_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n \qquad \text{provided} \quad p \neq q$$

Note:

$$u_0 = 0 \quad \text{so} \quad A_1 + A_2 = 0$$

Likewise:

$$u_l = 1 \quad \text{so} \quad A_1 + A_2\left(\frac{q}{p}\right)^l = 1$$

so:

$$-A_2 + A_2\left(\frac{q}{p}\right)^l = 1 \qquad \text{and thus} \qquad 1 = A_2\left[-1 + \left(\frac{q}{p}\right)^l\right] \qquad \text{giving} \qquad A_2 = \frac{1}{\left(\frac{q}{p}\right)^l - 1}$$

and:

$$A_1 = -A_2 = \frac{-1}{\left(\frac{q}{p}\right)^l - 1}$$

In consequence:

$$u_n = \frac{-1}{\left(\frac{q}{p}\right)^l - 1} + \frac{\left(\frac{q}{p}\right)^n}{\left(\frac{q}{p}\right)^l - 1}$$

giving:

$$u_n = \frac{\left(\frac{q}{p}\right)^n - 1}{\left(\frac{q}{p}\right)^l - 1}$$

as the final solution.

Observations about the solution:

First, $u_0 = 0$ and $u_l = 1$ as required.

Secondly, suppose $0 \ll n \ll l$ (e.g.: $l = 57$ and $n = 41$)...

If $\frac{q}{p} < 1$ [when $\left(\frac{q}{p}\right)^i \to 0$ for large i] the solution $u_n \to \frac{0-1}{0-1} \to 1$.

If $\frac{q}{p} > 1$ the solution $u_n \to \frac{(\frac{q}{p})^n\left[1-(\frac{p}{q})^n\right]}{(\frac{q}{p})^l\left[1-(\frac{p}{q})^l\right]} \to \frac{1}{(\frac{q}{p})^{l-n}}\left[\frac{1-0}{1-0}\right] \to 0$

In simple terms, provided n is well clear of the extremes 0 and l, u_n will tend to 1 or to 0 depending on whether $q < p$ or $q > p$. (It has been assumed that $p \neq q$.)

What about the case $p = q$ (as for an even coin)?

Recall that $w_1 = 1$ and $w_2 = \frac{q}{p}$ so the case $p = q$ implies *twin roots*, $w_1 = w_2 = 1$.
The general solution $u_n = A_1 w_1^n + A_2 w_2^n$ would be $u_n = A_1 + A_2$ which is silly.
In such cases, try a different guess:

$$u_n = (A_1 + A_2 n) w^n \qquad \text{where } w \text{ is the twin root}$$

In the present case, try:

$$u_n = (A_1 + A_2 n)\,(1)^n \tag{7.5}$$

as the general solution.

Check by substituting into $p\,u_{n+1} - u_n + q\,u_{n-1}$ thus:

$$p\,[A_1 + A_2(n+1)] - [A_1 + A_2 n] + q\,[A_1 + A_2(n-1)]$$

This, rearranged, is:

$$A_1[p - 1 + q] + A_2[pn + p - n + qn - q]$$

which, remembering that $p + q = 1$, is zero.

The next step is to determine values for A_1 and A_2 in the general solution whose revised form is:

$$u_n = (A_1 + A_2 n)\,(1)^n$$

Note:

$$u_0 = 0 \quad \text{so} \quad A_1 = 0$$

Likewise:

$$u_l = 1 \quad \text{so} \quad 0 + A_2 l = 1 \qquad \text{giving} \qquad A_2 = \frac{1}{l}$$

In consequence:

$$u_n = 0 + \frac{1}{l}\,n$$

giving:

$$u_n = \frac{n}{l}$$

as the final solution when the special case $p = q$ applies.

Second Order Inhomogeneous Linear Difference Equation

To solve:

$$v_n = 1 + p\,v_{n+1} + q\,v_{n-1} \qquad \text{given that} \quad v_0 = v_l = 0 \quad \text{and} \quad p + q = 1$$

Transfer all the terms except the 1 to the left-hand side:

$$p\,v_{n+1} - v_n + q\,v_{n-1} = -1$$

If the right-hand side were zero, this would be identical to the homogeneous equation just discussed. The new equation is solved in two steps. First, deem the right-hand side to be zero and solve as for the homogeneous case:

$$v_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n \qquad \text{provided} \quad p \neq q$$

Then, augment this solution by some $f(n)$ which has to be given further thought:

$$v_n = A_1(1)^n + A_2\left(\frac{q}{p}\right)^n + f(n)$$

This augmented v_n has to be such that when substituted into $p\,v_{n+1} - v_n + q\,v_{n-1}$ the result is -1.

From previous experience with u_n, it is known that substituting $A_1(1)^n + A_2\left(\frac{q}{p}\right)^n$ gives a result of zero. In consequence, the property required of $f(n)$ is that on substituting it into $p\,v_{n+1} - v_n + q\,v_{n-1}$ the result must be -1.

In this course, it will always be possible to express $f(n)$ as the quadratic $a + bn + cn^2$ with only one of the constants a, b and c non-zero. In the present case try $f(n) = k\,n$ and therefore require:

$$p\,k(n+1) - k\,n + q\,k(n-1) = -1$$

so:

$$p\,k\,n + p\,k - k\,n + q\,k\,n - q\,k = -1$$

Hence $\quad (p - q)\,k = -1 \quad$ so $\quad k = \frac{1}{q-p}$

giving:

$$v_n = A_1 + A_2\left(\frac{q}{p}\right)^n + \frac{n}{q-p} \tag{7.6}$$

as the general solution appropriate to the inhomogeneous difference equation. It is left as an exercise for the reader to determine values for A_1 and A_2 appropriate for the initial conditions given.

What about the case $p = q$?
When $p = q$ the equation:

$$p\,v_{n+1} - v_n + q\,v_{n-1} = -1$$

can be solved in two steps as before. First, deem the right-hand side to be zero and solve as for the homogeneous case:

$$v_n = (A_1 + A_2 n)(1)^n$$

Then, augment this solution by some $f(n)$ which has to be given further thought:

$$v_n = (A_1 + A_2 n)(1)^n + f(n)$$

As before, this augmented v_n has to be such that when substituted into $p\,v_{n+1} - v_n + q\,v_{n-1}$ the result is -1 but remember that $p = q$ this time.
Again, from previous experience with u_n, it is known that substituting $(A_1 + A_2 n)(1)^n$ gives a result of zero. Once more, the property required of $f(n)$ is that on substituting it into $p\,v_{n+1} - v_n + q\,v_{n-1}$ the result must be -1.

Since $p = q$, it is no use this time employing the previous approach which was to try $f(n) = k\,n$ and derive $k = \frac{1}{q-p}$. This is not a helpful value for k!
The appropriate approach now is to try $f(n) = k\,n^2$ and require:

$$p\,k(n+1)^2 - k\,n^2 + q\,k(n-1)^2 = -1$$

so:

$$p\,k\,n^2 + 2p\,k\,n + p\,k - k\,n^2 + q\,k\,n^2 - 2q\,k\,n + q\,k = -1$$

Hence $(p+q)\,k = -1$ so $k = -1$
giving:

$$v_n = A_1 + A_2 n - n^2 \qquad (7.7)$$

as the general solution appropriate to the inhomogeneous difference equation when $p = q$. Note that $A_1 + A_2 n$ is the solution to the homogeneous equation when $p = q$ and $-n^2$ is the required augmentation.
Given the initial conditions $v_0 = v_l = 0$, it is easy to determine that $A_1 = 0$ and $A_2 = l$ giving:

$$v_n = n\,(l - n)$$

as the final solution when the special case $p = q$ applies.

Glossary

The following technical terms have been introduced:

difference equations
order
linear
non-linear
homogeneous
inhomogeneous
initial condition
auxiliary equation
twin roots

Exercises — VII

When solving the inhomogeneous difference equations presented in problems 1, 6 and 7, recall that the function $f(n)$ can, in this course, always be expressed as the quadratic $a+bn+cn^2$ with only one of the constants a, b and c non-zero. You have seen as examples $f(n) = kn$ and $f(n) = kn^2$ and you should be prepared to try $f(n) = k$ on occasions.

1. Solve the linear first order inhomogeneous difference equations given as (7.1) from first principles.

2. Noting the expression for u_n given in (7.3), check that the first six values really are 1, 1, 2, 3, 5, 8.

3. Determine the values of the constants A_1 and A_2 in (7.4) given $u_0 = 1$ and $u_l = 0$.

4. Determine the values of the constants A_1 and A_2 in (7.5) given $u_0 = 1$ and $u_l = 0$.

5. Determine the values of the constants A_1 and A_2 in (7.6) given $v_0 = v_l = 0$.

6. Solve the following inhomogeneous equation:

$$u_{n+1} - u_n(1 - \alpha - \beta) = \alpha \qquad \text{given that} \quad u_0 = 0$$

   Note that α and β are constants in the range 0 to 1.

7. Solve the following inhomogeneous equation in which p is some probability:

$$2u_n + (1 - 2p)u_{n-1} = 1 \qquad \text{given that} \quad u_0 = 0$$