Introduction to Functional Programming

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Texts

Main Text:

Paulson, L.C. (1996). *ML for the Working Programmer*. Cambridge University Press (2nd ed.).

Other Useful Reading:

Backus, J. (1978). Can programming be liberated from the von Neumann style? *Communications of the ACM*, vol. 21, pp. 613-641.

Barendregt, H.P. (1984). *The Lambda Calculus: its Syntax and Semantics*. North-Holland.

Landin, P.J. (1966). The next 700 programming languages. *Communications of the ACM*, vol. 9, pp. 157-166.

Notes and exercises available from:

www.cl.cam.ac.uk/Teaching/2001/IntroFuncProg/

Imperative and Declarative

In an imperative programming language, the program provides a series of instructions (or commands) to the machine.

Examples of such languages include
C, Pascal, Modula2, Java

In a declarative programming language, the program (in principle) describes the computational task.

Functional: ML, Scheme, Haskell,. . .
Logic: Prolog, Godel,. . .

Programming Views

Imperative languages present a level of abstraction above the machine, hiding some details (memory addresses, registers, etc.)

Still, the view is machine-centred.

Declarative languages provide a still further level of abstraction.

A style of programming that is more programmer-centred.

Functional programming

In the functional programming style, the computational task to be programmed is taken to be a function (in the mathematical sense).

The job of the programmer is to describe this function.

Implicit in the description is a method for computing the function.

The function maps one domain (of inputs) to another (of outputs).

These may be: integers; real numbers; lists; strings; or even functions themselves

importance of types

Commands and Expressions

In a typical imperative language, commands are formed from assignments to variables:

$$x := E$$

by application of various control structures.

Sequencing

$$C_1; C_2$$

Conditionals

$$\texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2$$

Looping

$$\texttt{while}\ B\ \texttt{do}\ C$$

Expressions

A functional program is just an expression to be evaluated.

An expression is built up from simpler expressions by means of function applications.

$$E_1 + E_2$$

or

$$\texttt{if}\ B\ \texttt{then}\ E_1\ \texttt{else}\ E_2$$

There are no explicit notions of variable assignment, sequencing or control.

Example: the factorial

The factorial function can be written imperatively in C as follows:

```
int fact(int n)
{ int x = 1;
  while (n > 0)
   { x = x * n;
     n = n - 1;
   }
  return x;
}
```

whereas it would be expressed in ML as a recursive function:

```
fun fact n =
  if n = 0 then 1
  else n * fact(n - 1);
```

Recursion

Recursive definition of functions is crucial to functional programming.

There is no other mechanism for looping

Variables cannot be updated through assignment. They get their values from function calls.

Type Checking

ML provides type checking, which can help catch many programming errors.

Types in ML may be polymorphic.

```
fun length []      = 0
  | length (x::l) = 1 + length (l);
```

Advantages

"Attack complexity with simple abstractions"

- Clarity
- Expressiveness
- Shorter Programs
- Security through type system
- Ease of reasoning
- Better modularity

Disadvantages

- Input/Output
- Interactivity and continuously running programs
- Speed/Efficiency

There is no reasonable "pure" functional language

Brief History

- Lambda Calculus (Church 1936)
- LISP (McCarthy 1954)
- ISWIM (Landin 1966)
- ML (Milner et al., 1974), originally a Meta Language for the LCF Theorem Prover.
- Definition of Standard ML (Milner, Tofte and Harper 1990)
- Revised definition and standard library (1997)

Rest of the Course

11 more lectures covering

- Basic Types in Standard ML
- Lists and Recursion
- Sorting
- Datatypes
- Higher Order Functions
- Specification and Verification
- Types and Type Inference
- Substantial case study

Running ML

ML provides an interactive session.

Enter an expression. ML returns a value.

```
Moscow ML version 1.42 (July 1997)
Enter ‘quit();’ to quit.
- (2*4) + 18;
> val it = 26 : int
- 2.0 * 2.0 * 3.14159;
> val it = 12.56636 : real
```

```
- val pi = 3.14159;
> val pi = 3.14159 : real
- val a = pi* 2.0 *2.0;
> val a = 12.56636 : real
- val a = 2 * pi;

...
! Type clash: expression of type
!   real
! cannot have type
!   int

- val area = fn r => pi*r*r;
> val area = fn : real -> real
- val sqr = fn r => r*r;
> val sqr = fn : int -> int
- val sqr = fn r:real => r*r;
> val sqr = fn : real -> real
- val sqr = fn r => r*r:real;
> val sqr = fn : real -> real
```

```
- fun area (r) = pi*r*r;
> val area = fn : real -> real
- val pi = "yabadabadoo";
> val pi = "yabadabadoo" : string
- area(2.0);
> val it = 12.56636 : real
- area;
> val it = fn : real -> real
- it(2.0);
> val it = 12.56636 : real
- area(2);


...
! Type clash: expression of type
!   int
! cannot have type
!   real
```

Numeric Types

`int`: the integers

- constants `0 1 ~1 2 ~2 0032...`
- infix operators `+ - * div mod`

`real`: the floating point numbers

- constants `0.0 ~1.414 2.0 3.94e~7 ...`
- infix operators `+ - * /`

Overloading

Functions defined for both `int` and `real`:

- operators `~ + - *`
- relations `< <= > >=`

You must tell the type checker what type is intended, if there is any ambiguity.

Basis Library

Useful library of functions, collected together into structures.

```
Int    Real    Math
```

The basis library is automatically loaded when using SML/NJ.

May need to be explicitly loaded in Moscow ML.

```
>- load "Math";
> val it = () : unit
- fun f u = Math.sin(u)/u;
> val f = fn : real -> real
```

To load your own file of definitions:

```
- use "myfile";
```

Strings

Type `string`

- constants `"" "A" "yaba!!daba&doo\n"`
- `size: string -> int`
  determines the number of characters in a string.
- `s1^s2`
  the concatenation of strings `s1` and `s2`
- relations `< <= > >=`

Structure `String`

Characters

Type `char`

- constants `#"A" #"y" #" "`
- `ord: char -> int` integer value of a character.
- `chr: int -> char`
- relations `< <= > >=`

Structure `Char`

Truth Values

Type `bool`

- constants `true false`
- `not: bool -> bool`
- `if p then x else y`

p `andalso` q

`if` p `then` q `else false`

p `orelse` q

`if` p `then true else` q

Structure `Bool`

Pairs and Tuples

```
- (2,3);
> val it = (2, 3) : int * int

- (2.0,2,3,"aa");

> val it = (2.0, 2, 3, "aa") :
real * int * int * string
```

Tuples are useful for representing vectors, presenting functions with multiple arguments, obtaining multiple results from a function, etc.

```
 - fun addtwice (m,n) = m + 2*n;
> val addtwice = fn : int * int -> int
```

Vectors

```
- fun negvec(x,y):real*real = (~x,~y);

> val negvec =
fn : real * real -> real * real

- negvec(1.0,1.0);
> val it = (~1.0, ~1.0) : real * real

- fun addvec((x1,y1),(x2,y2)):real*real =
  (x1+x2,y1+y2);

> val addvec = fn : (real * real) *
        (real * real) -> real * real

- fun subvec(v1,v2) = addvec(v1,negvec v2);


> val subvec = fn : (real * real) *
      (real * real) -> real * real
```

Evaluation Strategy

Strict (or eager) evaluation.

Also known as call-by-value

Given an expression, which is a function application

$$f(E_1, \ldots, E_n)$$

evaluate $E_1, \ldots, E_n$ and then apply f to the resulting values.

Call-by-name:

Substitute the expressions $E_1, \ldots, E_n$ into the definition of f and then evaluate the resulting expression.

Lazy Evaluation

Also known as call-by-need.

Like call-by-name, but sub-expressions that appear more than once are not copied. Pointers are used instead.

Potentially more efficient, but difficult to implement.

Standard ML uses strict evaluation.

Lists

A list is an ordered collection (of any length) of elements of the same type

```
- [1,2,4];
> val it = [1, 2, 4] : int list
- ["a" , "", "abc", "a"];
> val it =  . . .  : string list
- [[1],[],[2,3]];
> val it = . . . : int list list
- [];
> val it = [] : 'a list
- 1::[2,3];
> val it = [1, 2, 3] : int list
```

Lists

There are two kinds of list:

`nil` or `[]` is the empty list

`h::t` is the list with head `h` and tail `t`

`::` is an infix operator of type

```
fn : 'a * 'a list -> 'a list
```

$[x_1, \ldots, x_n]$ is shorthand for

$$x_1\texttt{::}(\cdots(x_n\texttt{::}\texttt{nil})\cdots)$$

Built-in Functions 1

`null`

```
fn : 'a list -> bool
```

determines if a list is empty

`hd`

```
fn : 'a list -> 'a
```

gives the first element of the list

`tl`

```
fn : 'a list -> 'a list
```

gives the tail of the list

Built-in Functions 2

`length`

```
fn : 'a list -> int
```

gives the number of elements in a list

`rev`

```
fn : 'a list -> 'a list
```

gives the list in reverse order

`@`

appends two lists NB: infix!

List Functions

```
fun null l =
     if l = [] then true else false;
```

or, using pattern matching:

```
fun null []      = true
  | null (_::_) = false;

fun hd (x::_) = x;

fun tl (_::l) = l;
```

NB: these functions are built-in and do not need to be defined

Recursive definitions

```
fun rlength []      = 0
  | rlength (h::t) = 1 + rlength(t);


fun append ([], l)   = l
  | append (h::t, l) = h::append(t,l);


fun reverse []      = []
  | reverse (h::t) = reverse(t)@[h];
```

Purely recursive definitions can be very inefficient

Iterative Definitions

```
fun addlen ([],n)     = n
  | addlen (h::t, n) = addlen (t, n+1);
```

```
fn : 'a list * int -> int
```

```
fun ilength l = addlen(l,0);
```

```
fun revto ([],l)     = l
  | revto (h::t, l) = revto (t, h::l);
```

```
fn : 'a list * 'a list -> 'a list
```

Library List Functions

```
load "List";
```

We can then use `List.take`, `List.drop`

```
fun take (k, [])    = []
  | take (k, h::t) =
      if k > 0 then h::take(k-1,t)
               else [];

fun drop (k, [])    = []
  | drop (k, h::t) =
      if k > 0 then drop(k-1,t)
               else h::t;
```

```
fn : int * 'a list -> 'a list
```

List of Pairs

```
fun zip ([], []) = []
  | zip (h1::t1,h2::t2) =
            (h1,h2)::zip(t1,t2);
```

```
! Warning: pattern matching is not
            exhaustive

> val zip = fn :
     'a list * 'b list -> ('a * 'b) list
```

Creates a list of pairs from a pair of lists.

What happens when the two lists are of different length?

Unzipping

```
fun unzip []                = ([],[])
  | unzip ((x,y)::pairs) =
      let val (t,u) = unzip pairs in
        (x::t, y::u)
        end;
```

Note the local declaration

let D in E end

Compare this against applying functions `first` and `second` to extract the components of the pair.

Equality Types

We can test certain expressions for equality:

```
- 2 = 1+1;
> val it = true : bool
- 1.414*1.414 = 2.0;
> val it = false : bool
- [] = [1];
> val it = false : bool
```

Equality testing can be used with the basic types, and with tuples and lists, *but not with functions.*

```
- (fn x => x+2) = (fn x => 2+x);

! Type clash: match rule of type
!   'a -> 'b
! cannot have equality type ''c
```

Testing for Membership

```
fun member (x, [])    = false
  | member (x, h::t) =
        (x=h) orelse member (x,t);
```

```
val member = fn : ''a * ''a list -> bool
```

''a is an *equality type variable*.

```
- op=;
> val it = fn : ''a * ''a -> bool
```

```
fun inter ([], l)  = []
  | inter (h::t,l) =
        if member (h,l) then h::inter(t,l)
                        else inter(t,l);
```

```
fn : ''a list * ''a list -> ''a list
```

Insertion Sort

```
fun insert(x:real, []) = [x]
  | insert(x, h::t)    =
        if x <= h then x::(h::t)
                  else h::insert(x,t);
```

```
fun insort []      = []
  | insort (h::t) = insert (h, insort t);
```

```
fn : real list -> real list
```

Insertion sort takes $O(n^2)$ comparisons on average and in the worst case.

Merge Sort

```
fun merge ([], l)          = l : real list
  | merge (l, [])          = l
  | merge (h1::t1,h2::t2)=
      if h1 <= h2
          then h1::merge(t1, h2::t2)
          else h2::merge(h1::t1, t2);


fun mergesort []     = []
  | mergesort [x] = [x]
  | mergesort l    =
     let val k = length l div 2 in
      merge(mergesort (List.take(l, k)),
            mergesort (List.drop(l, k)))
     end;
```

Merge sort takes $O(n \log n)$ comparisons on average and in the worst case.

Quick Sort

```
fun quick []      = []
  | quick [x]     = [x] : real list
  | quick (h::t) =
 let fun part (left, right, []) =
          (quick left)@(h::quick right)
       | part (left, right, x::l) =
          if x<=h
            then part (x::left, right, l)
            else part (left, x::right, l)
 in
   part( [], [], t) end;
```

Quick sort takes $O(n \log n)$ comparisons on average and $O(n^2)$ in the worst case.

QS without Append

```
fun quik ([], sorted)    = sorted
  | quik ([x], sorted)   = (x:real)::sorted
  | quik (h::t, sorted) =
 let
  fun part (left, right, [])  =
       quik(left, h::quik(right, sorted))
    | part (left, right, x::l) =
       if x<= h
          then part (x::left, right, l)
          else part (left, x::right, l)
 in
   part([], [], t) end;
```

Record Types

```
- { name="Jones", salary=20300, age=26};

val it =
{age = 26, name = "Jones", salary = 20300}
: {age : int, name : string, salary : int}


- {1="Jones", 2=20300,3=26};

> val it = ("Jones", 20300, 26)
           : string * int * int
```

Record Pattern Matching

```
- val emp1 =
{name="Jones", salary=20300, age=26};

> val emp1 =
{age = 26, name = "Jones", salary = 20300}
: {age : int, name : string, salary : int}

- val {name=n1,salary=s1,age=a1}= emp1;

        > val n1 = "Jones" : string
          val s1 = 20300 : int
          val a1 = 26 : int

- val {name=n1,salary=s1,...} = emp1;

        > val n1 = "Jones" : string
          val s1 = 20300 : int

- val {name,age,...} = emp1;

        > val name = "Jones" : string
          val age = 26 : int
```

Record Types

```
type employee = {name: string,
                 salary: int,
                 age: int};
```

```
> type employee = ...
```

```
fun tax (e: employee) =
         real(#salary e)*0.22;
```

Or,

```
fun tax ({salary,...}: employee) =
       real(salary)*0.22;
```

Enumerated Types

Consider the King and his court:

```
datatype degree = Duke
                | Marquis
                | Earl
                | Viscount
                | Baron;

datatype person =
            King
          | Peer of degree*string*int
          | Knight of string
          | Peasant of string;
```

All constructors are distinct.

Functions on Datatypes

```
[King,
 Peer(Duke, "Gloucester", 5),
 Knight "Gawain",
 Peasant "Jack Cade"];
```

```
val it  = ... : person list
```

```
fun superior (King, Peer _) = true
  | superior (King, Knight _) = true
  | superior (King, Peasant _) = true
  | superior (Peer _,Knight _) = true
  | superior (Peer _, Peasant _) = true
  | superior (Knight _, Peasant _) = true
  | superior _ = false;
```

Exceptions

Exceptions are raised when there is no matching pattern, when an overflow occurs, when a subscript is out of range, or some other run-time error occurs.

Exceptions can also be explicitly raised.

```
exception Failure;
exception BadVal of Int;

raise Failure
raise (BadVal 5)
```

E `handle` P_1 `=>` E_1 `|` ... `|` P_n `=>` E_n

Recursive Datatypes

The built-in type operator of lists might be defined as follows:

```
infix :: ;

datatype 'a list = nil
                 | :: of 'a * 'a list;
```

Binary Trees:

```
datatype 'a tree =
            Lf
          | Br of 'a * 'a tree * 'a tree;
```

```
Br(1, Br(2, Br(4, Lf, Lf),
            Br(5, Lf, Lf)),
      Br(3, Lf, Lf))
```

Functions on Trees

Counting the number of branch nodes

```
fun count Lf              = 0
  | count (Br(v,t1,t2)) =
              1+count(t1)+count(t2);
```

```
val count = fn : 'a tree -> int
```

Depth of a tree

```
fun depth Lf              = 0
  | depth (Br(v,t1,t2)) =
              1+Int.max(depth t1, depth t2);
```

```
val depth = fn : 'a tree -> int
```

Listing a Tree

Three different ways to list the data elements of a tree

Pre-Order

```
fun preorder Lf            = []
  | preorder (Br(v,t1,t2))=
      [v] @ preorder t1 @ preorder t2;
```

In-Order

```
fun inorder Lf             = []
  | inorder (Br(v,t1,t2))=
      inorder t1 @ [v] @ inorder t2;
```

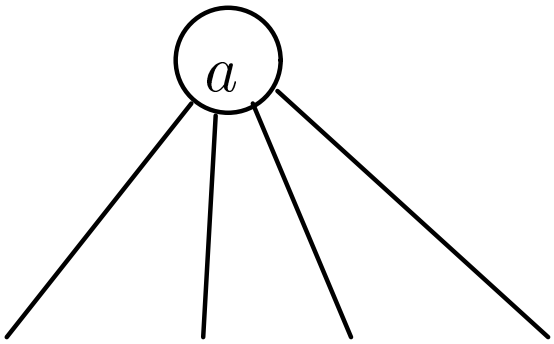
Post-Order

```
fun postorder Lf           = []
  | postorder (Br(v,t1,t2))=
      postorder t1 @ postorder t2 @ [v];
```

Multi-Branching Trees

To define a datatype of a tree where each node can have any number of children

```
datatype 'a mtree =
        Branch of 'a * ('a mtree) list;
```



To recursively define functions, we can use `map`.

```
fun double (Branch(k,ts)) =
          Branch(2*k, map double ts);
```

Arrays

Arrays in an imperative language designate a contiguous block of memory locations.

An array can be updated in place.

```
A[k] := x
```

There are no such arrays in a purely functional language.

A functional array can be thought of as a function from a finite set of integers.

$$A : \{1, \ldots, k\} \to \alpha$$

We require an update operation:

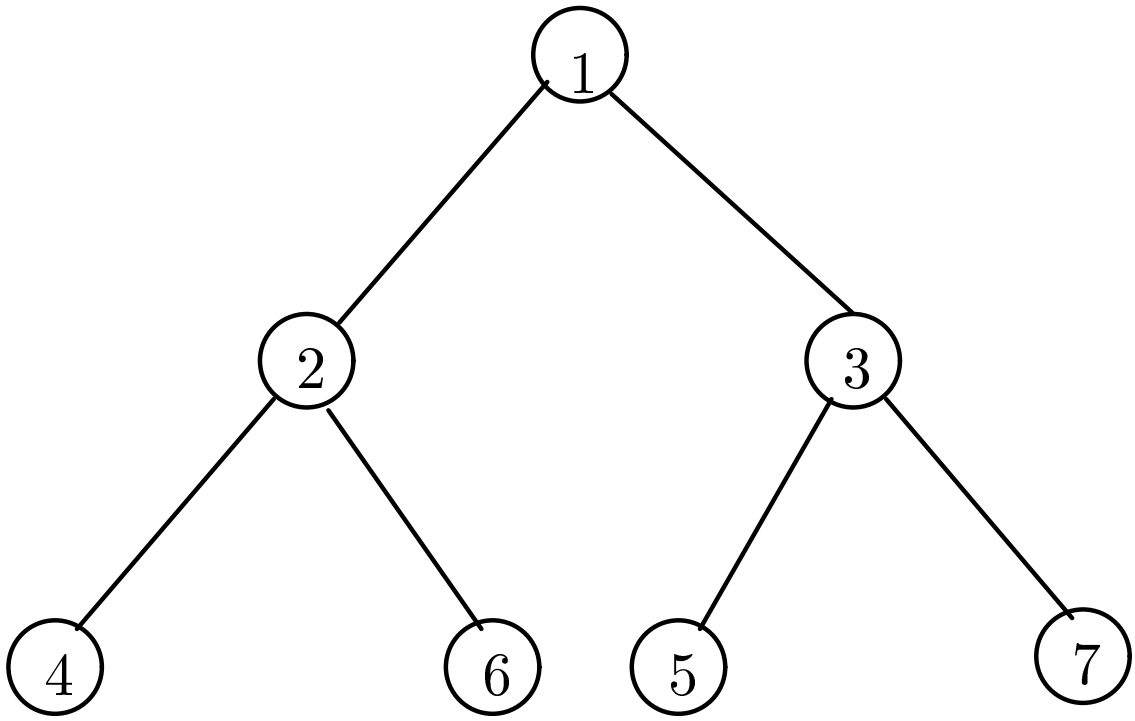
```
B = update(A, k, x)
```

This creates a new copy of the array, which is the same as A except for the value at k.

Arrays as Binary Trees

A tree whose depth is equal to the maximum number of bits in a subscript

For a subscript k, take its binary representation, and at each level, follow the left branch if the bit is 0 and the right branch if it is 1.



Lookup

```
exception Array;     (*out of range*)

fun asub (Lf, _)            = raise Array
  | asub (Br(v,t1,t2), k) =
        if       k = 1
                  then v
        else if k mod 2 = 0
                  then asub(t1, k div 2)
                  else asub(t2, k div 2);
```

```
val asub = fn : 'a tree * int -> 'a
```

For an array A and integer k `asub(A,k)` gives the value of $A[k]$.

Update

```
fun aupdate (Lf, k, w)          =
     if k = 1 then Br(w, Lf, Lf)
                 else raise Array
  | aupdate (Br(v,t1,t2), k, w) =
     if      k = 1
      then
       Br(w, t1, t2)
     else if k mod 2 = 0
      then
       Br(v,aupdate(t1,k div 2,w),t2)
     else
      Br(v,t1,aupdate(t2,k div 2,w));
```

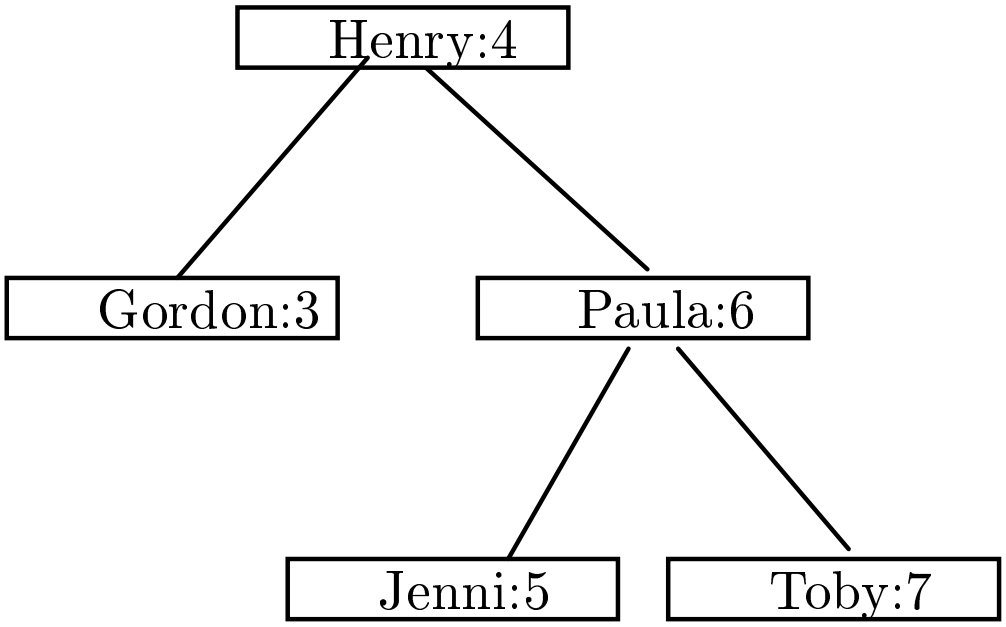
```
val aupdate =
 fn : 'a tree * int * 'a -> 'a tree
```

Binary Tree Directories

A *Directory* associates *values* with *keys*.

If a directory is implemented as a list, each update and each lookup takes $O(n)$ operations *on the average*.

When the directory is implemented as a binary search tree, the operations take $O(\log n)$ on the average.



Search Tree Lookup

```
exception Bsearch of string;

fun blookup (Lf, b) = raise Bsearch(b)

  | blookup (Br((a,x), t1, t2), b)  =
     if      b<a then blookup(t1, b)
     else if a<b then blookup(t2, b)
     else x;


exn Bsearch = fn : string -> exn

val blookup =
fn : (string * 'a) tree * string -> 'a
```

Update

```
fun bupdate (Lf, k:string, v) =
      Br((k,v), Lf, Lf)
  | bupdate (Br((a,x),t1,t2), k, v) =
      if      k < a
        then Br((a,x),bupdate(t1,k,v),t2)
      else if a < k
        then Br((a,x),t1,bupdate(t2,k,v))
      else   Br((k,v),t1,t2);    (*a=k*)
```

```
val bupdate =
fn : (string * 'a) tree * string * 'a
     -> (string * 'a) tree
```

Higher-Order Functions

A higher order function (also called a *functional*) is a function that either takes a function as an argument, or yields a function as a result.

Higher order functions are a key feature that distinguishes functional from imperative programming.

- partial evaluation
- general purpose functionals (such as `map`)
- sequences or infinite lists
- search strategies

Function Values

The expression

$$\texttt{fn}\ x\ \texttt{=>}\ E$$

is a function-valued expression.

Pattern-matching can be used:

$$\texttt{fn}\ P_1\ \texttt{=>}\ E_1\ \texttt{|}\ \ldots\ \texttt{|}\ P_n\ \texttt{=>}\ E_n$$

The declaration

```
val double = fn n=>n*n;
```

is the same as:

```
fun double n = n*n;
```

$$\texttt{if}\ E\ \texttt{then}\ E_1\ \texttt{else}\ E_2$$

is defined as:

```
(fn true => E1 | false => E2) E
```

Curried Functions

Every function in ML takes *only one* argument.

For functions that require two arguments, we can use pairs:

```
- op+ (5,4);
> val it = 9 : int
- op+;
> val it = fn : int * int -> int
```

Or we can define:

```
- fun plus m n = m+n;
> val plus = fn : int -> int -> int
```

This is the *curried* version of `plus`

Partial evaluation:

```
- map (plus 4) [2, 3, 4];
> val it = [6, 7, 8] : int list
```

More Curried Functions

```
- hd;
> val it = fn : 'a list -> 'a

- hd [op+,op-,op*,op div] (5,4);
> val it = 9 : int
```

Here the type of `hd` is:

```
(int*int -> int) list -> int*int -> int
```

An analogy can be made with nested arrays, as in Pascal:

```
A: array [1..10] of
    array [1..10] of real

              . . .A[i][j]. . .
```

Generic Sorting

```
fun insort lessequal =
 let fun ins (x,[])  = [x]
       | ins (x,h::t)=
          if lessequal(x,h) then x::h::t
          else h::ins(x,t)
     fun sort [] = []
       | sort (x::l) = ins(x,sort l)
  in sort end;

> val insort = fn :
  ('a * 'a -> bool) ->
     ('a list -> 'a list)

- insort (op<=) [5,3,5,7,2,9];
> val it = [2, 3, 5, 5, 7, 9] : int list
- insort (op>=) [5,3,5,7,2,9];
> val it = [9, 7, 5, 5, 3, 2] : int list
```

A Summation Functional

```
fun sum f 0 = 0.0
  | sum f m = f(m-1) + sum f (m-1);

> val sum =
  fn : (int -> real) -> int -> real
```

$$\texttt{sum f m} = \sum_{i=0}^{m-1} f(i)$$

$$\texttt{sum (sum f) m} = \sum_{i=0}^{m-1} \sum_{j=0}^{i-1} f(j)$$

Matrix Transpose

The `map` functional applies a function to every element of a list

```
fun map f []      = []
  | map f (h::t) = (f h)::(map f t);
```

Representing a matrix as a list of lists, the following defines the transpose function.

```
fun transp ([]::_) = []
  | transp rows    =
              (map hd rows)::
              (transp (map tl rows));
```

```
fn : 'a list list -> 'a list list
```

Matrix Multiplication

The dot product of two vectors as a curried function:

```
fun dotprod [] []                = 0.0
  | dotprod (h1::t1) (h2::t2) =
      h1*h2 + dotprod t1 t2;
```

Matrix multiplication:

```
fun matmult (Arows, Brows) =
 let val cols = transp Brows
 in map (fn row => map (dotprod row) cols)
      Arows
 end;
```

The Fold Functional

`foldl` and `foldr` are built-in functionals which can be defined as:

```
fun foldl f e []  = e
  | foldl f e (h::t) =
          foldl f (f(h,e)) t;

fun foldr f e []    = e
  | foldr f e (h::t) =
         f(h, foldr f e t);
```

These can be used to give simple definitions of many list functions

`foldl op+ 0`	sum
`foldl (fn (_,n) => n+1) 0`	length
`foldr op:: xs ys`	`ys@xs`

Predicates

```
fun exists p []      = false
  | exists p (h::t) = (p h) orelse
                          exists p t;
```

```
fn : ('a -> bool) -> 'a list -> bool
```

Determines whether there is any element in a list that satisfies the predicate `p`.

```
fun filter p []      = []
  | filter p (h::t) = if p h then
                          h::(filter p t)
                      else filter p t;
```

```
fn : ('a -> bool) -> 'a list -> 'a list
```

Lazy Lists

Lists where the next element is computed on demand.

Also known as *streams* in the literature.

(not to be confused with input/output streams)

- Avoids waste if the entire list is not needed.
- Infinite objects are a useful abstraction.
- Models simple I/O.

Disadvantage: Termination is lost

Lazy Lists in ML

`()` —the unique tuple of length 0 has type `unit`.

A datatype for lazy lists:

```
datatype 'a seq = Nil
  | Cons of 'a * (unit -> 'a seq);
```

`Cons`(h, tf) is the sequence with head h and tail function tf.

Example: the infinite sequence $k, k+1, k+2, \ldots$

```
fun from k =
  Cons(k, fn () => from(k+1));
```

```
val from = fn : int -> int seq
```

Operations on Lazy Lists

```
exception Empty;

fun hdq Nil = raise Empty
  | hdq (Cons(h,_)) = h;

fun tlq Nil = raise Empty
  | tlq (Cons(_,tf)) = tf();

from 1;
> val it = Cons(1, fn) : int seq
- tlq it;
> val it = Cons(2, fn) : int seq
- tlq it;
> val it = Cons(3, fn) : int seq
```

Consuming a Sequence

```
fun takeq (0,s)          = []
  | takeq (n,Nil)        = []
  | takeq (n,Cons(h,tf))=
            h::takeq(n-1,tf());
```

takeq takes an integer n and a sequence s and produces a list of the first n elements of s

```
fun squares Nil :int seq = Nil
  | squares (Cons(h,tf)) =
      Cons(h*h,fn () => squares (tf()));
```

```
- squares (from 1);
> val it = Cons(1, fn) : int seq
- takeq (5,it);
> val it = [1, 4, 9, 16, 25] : int list
```

Joining Two Sequences

```
fun appendq (Nil, yf)  = yf
  | appendq (Cons(h,tf),yf) =
      Cons(h, fn() => appendq(tf(),yf));
```

```
appendq: 'a seq * 'a seq -> 'a seq
```

If the first sequence is infinite,

```
appendq(x,y) = x
```

```
fun interleave (Nil, yf) = yf
  | interleave (Cons(h,tf),yf) =
      Cons(h,fn() => interleave(yf,tf()));
```

Functionals for Lazy Lists

```
fun mapq f Nil  = Nil
  | mapq f (Cons(h,tf)) =
      Cons(f h, fn() => mapq f (tf()));

- mapq (fn x => x*x) (from 1);
> val it = Cons(1, fn) : int seq
- takeq(5,it);
> val it = [1, 4, 9, 16, 25] : int list

fun filterq p Nil = Nil
  | filterq p (Cons(h,tf)) =
      if p h then
       Cons(h,fn() => filterq p (tf()))
      else filterq p (tf());
```

Searching Infinite Trees

Many problem solving applications involve searching a very large search space, structured as a tree.

Usually the tree is not available in its entirety, but the nodes are generated as needed.

We represent such a tree as a function of type

$$\texttt{next} : \alpha \rightarrow \alpha\ \texttt{list}$$

Different search strategies may be appropriate.

- *Depth First Search* searches the entire sub-tree rooted at a before proceeding to its sibling.
- *Breadth First Search* searches all nodes at a given level before proceeding to the next level.

Depth First Search

Fast, when it works

Requires space proportional to the *height* of the tree.

If the tree is infinite, it may not find a solution even if one exists.

```
fun depth next x =
 let fun dfs []     = Nil
       | dfs (h::t)=
          Cons(h,fn() => dfs((next h)@t))
 in dfs [x] end;
```

```
fn : ('a -> 'a list) -> 'a -> 'a seq
```

Breadth First Search

Guaranteed to find a solution in finite time, if one exists.

Finds the nearest solution first.

Requires space proportional to the *size* of the tree.

```
fun breadth next x =
 let fun bfs []     = Nil
       | bfs (h::t)=
          Cons(h,fn() => bfs(t@(next h)))
 in bfs [x] end;
```

```
fn : ('a -> 'a list) -> 'a -> 'a seq
```

Testing and Verification

We wish to establish that a program is correct.
That is to say, it meets its specifications.

Testing

Try a selection of inputs and check against results

There is no guarantee that all bugs will be found.
Behaviour of a program is not continuous

Verification

Prove that the program is correct.

Proofs can be long, tedious and complicated.
Functional programs are easier to do proofs with.

Formal vs. Rigorous proof

Formal Proof:

- what logicians study
- purely symbolic
- needs an axiom system
- needs machine support

Rigorous Proof:

- what mathematicians do
- in natural language (with some symbols)
- needs clear foundations
- understandable to people

A rigorous proof is a convincing mathematical argument.

Assumptions

In order to reason with ML programs, we make some assumptions about the expressions that will be used in proofs.

- Expressions are purely functional.
- Expressions are well-formed and well-typed.
- Types are interpreted as sets.
- Execution of programs always terminates.

```
fun undef(x) = undef(x + 1);
```

`factorial` terminates for $n \geq 0$.

Principle of Induction

We have a property $\phi(n)$ we wish to prove for all integers $n \geq 0$.

Prove:

- $\phi(0)$; and
- $\phi(k)$ implies $\phi(k+1)$, for all k.

This proves $\phi(n)$ for all n.

For any particular n, $\phi(n)$ can be derived in n steps.

Two Factorial Functions

```
fun fact n =
    if n = 0 then 1 else n*(fact n-1);
```

Or, a more efficient, iterative, version

```
fun facti (n,p) =
    if n=0 then p else
            facti(n-1, n*p);
```

Are the two functions equivalent?

Can we show that for all n:

$$\texttt{facti}(n, 1) = \texttt{fact}(n)$$

Inductive Proof

We make the stronger inductive hypothesis.

$$\forall p\ \texttt{facti}(n, p) = \texttt{fact}(n) \times p$$

Base case:

$$\begin{aligned} \texttt{facti}(0, p) &= p && \text{defn. of } \texttt{facti} \\ &= 1 \times p \\ &= \texttt{fact}(0) \times p && \text{defn. of } \texttt{fact} \end{aligned}$$

Induction Step:

$$\begin{aligned} \texttt{facti}(k+1, p) &= \texttt{facti}(k, (k+1) \times p) \\ &= \texttt{fact}(k) \times (k+1) \times p \\ &= \texttt{fact}(k+1) \times p \end{aligned}$$

Complete Induction

To show that $\phi(n)$ is true for all $n \geq 0$, it is also sufficient to show that:

$$(\forall i < k\, \phi(i)) \text{ implies } \phi(k)$$

- no separate base case is necessary
- $\phi(n)$ is still provable in n steps
- a more general form of proof by induction

Proof by Complete Induction

```
fun power(x,k) =
    if k=1 then x
    else if k mod 2 = 0 then
         power(x*x, k div 2)
    else x*power(x*x, k div 2);
```

Prove by induction (on k) that for all $k \geq 1$:

$$\forall x \; \texttt{power}(x, k) = x^k$$

Complete Induction (*contd.*)

if $k = 1$ then $\texttt{power}(x, 1) = x = x^1$

if $k = 2j$ then

$$\begin{aligned}\texttt{power}(x, k) &= \texttt{power}(x, 2j) \\ &= \texttt{power}(x^2, j) \\ &= (x^2)^j \\ &= x^{2j}\end{aligned}$$

if $k = 2j + 1$ then

$$\begin{aligned}\texttt{power}(x, k) &= \texttt{power}(x, 2j + 1) \\ &= x \times \texttt{power}(x^2, j) \\ &= x \times (x^2)^j \\ &= x^{2j+1}\end{aligned}$$

Structural Induction

To prove a property $\phi(l)$ for all lists l, it suffices to show:

- base case: $\phi([])$
- induction step: for any h and t, if $\phi(t)$, then $\phi(h :: t)$

$\phi([x_1, \ldots, x_n])$ can be proved in n steps.

Similarly for other recursive datatypes.

Some List Functions

```
fun app([], l)    = l
  | app(h::t, l) = h::app(t, l);


fun nrev []       = []
  | nrev (h::t) = (nrev t)@[h];


fun revto ([], l)    = l
  | revto (h::t, l) = revto (t,h::l);
```

We can prove properties such as:

$$\texttt{app}(\texttt{app}(l1, l2), l3) = \texttt{app}(l1, \texttt{app}(l2, l3))$$

$$\texttt{revto}(l1, l2) = \texttt{nrev}(l1)@l2$$

Append is Associative

For all $l1$, $l2$ and $l3$:

$$\mathtt{app}(\mathtt{app}(l1, l2), l3) = \mathtt{app}(l1, \mathtt{app}(l2, l3))$$

By structural induction on $l1$.

Base case:

$$\begin{aligned} \mathtt{app}(\mathtt{app}([], l2), l3) &= \mathtt{app}(l2, l3) \\ &= \mathtt{app}([], \mathtt{app}(l2, l3)) \end{aligned}$$

Induction Step:

$$\begin{aligned} \mathtt{app}(\mathtt{app}(h :: t, l2), l3) &= \mathtt{app}(h :: \mathtt{app}(t, l2), l3) \\ &= h :: \mathtt{app}(\mathtt{app}(t, l2), l3) \\ &= h :: \mathtt{app}(t, \mathtt{app}(l2, l3)) \\ &= \mathtt{app}(h :: t, \mathtt{app}(l2, l3)) \end{aligned}$$

Reverse

$$\forall\, l2\ \texttt{revto}(l1, l2) = \texttt{nrev}(l1)@l2$$

By structural induction on $l1$.

Base case:

$$\texttt{revto}([], l2) = l2 = []@l2 = \texttt{nrev}[]@l2$$

Induction Step:

$$\begin{aligned}
\texttt{revto}(h :: t, l2) &= \texttt{revto}(t, h :: l2) \\
&= \texttt{nrev}(t)@(h :: l2) \\
&= \texttt{nrev}(t)@[h]@l2 \\
&= \texttt{nrev}(h :: t)@l2
\end{aligned}$$

Other Examples

$$
\begin{aligned}
\texttt{nlength}(l1@l2) &= (\texttt{nlength}\, l1) + (\texttt{nlength}\, l2) \\
\texttt{nrev}(l1@l2) &= (\texttt{nrev}\, l2)@(\texttt{nrev}\, l1) \\
\texttt{nrev}(\texttt{nrev}\, l) &= l \\
l@[] &= l \\
(\texttt{map}\, f) \circ (\texttt{map}\, g) &= \texttt{map}(f \circ g)
\end{aligned}
$$

Correctness preserving program transformations.

Structural Induction on Trees

```
datatype 'a tree =
          Lf
        | Br of 'a*'a tree*'a tree
```

To show $\phi(t)$ for all trees t, it suffices to show:

- $\phi(\texttt{Lf})$ Base case
- For any t_1, t_2 and x:
  if $\phi(t_1)$ and $\phi(t_2)$ then $\phi(\texttt{Br}(x, t_1, t_2))$.
  Induction step

$\phi(t)$ can be proved in $\texttt{size}(t)$ steps.

Two Preorder Functions

```
fun preorder Lf              = []
  | preorder (Br(x,t1,t2)) =
      x::preorder(t1)@preorder(t2);
```

```
fun preord (Lf, l)           = l
  | preord (Br(x,t1,t2),l) =
      x::preord(t1, preord (t2,l));
```

We can show that the two are equivalent:

$$\forall\, l\ \texttt{preord}(t, l) = \texttt{preorder}(t)@l$$

By induction on t

Preorder

Base Case:

$$\texttt{preord}(\texttt{Lf}, l) = l = []@l = \texttt{preorder}(\texttt{Lf})@l$$

Induction Step:

$$\begin{aligned}
&\texttt{preord}(\texttt{Br}(x, t_1, t_2), l) \\
&\qquad = x :: \texttt{preord}(t_1, \texttt{preord}(t_2, l)) \\
&\qquad = x :: \texttt{preorder}(t_1)@\texttt{preord}(t_2, l) \\
&\qquad = x :: \texttt{preorder}(t_1)@\texttt{preorder}(t_2)@l \\
&\qquad = \texttt{preorder}(\texttt{Br}(x, t_1, t_2))@l
\end{aligned}$$

Induction Principles

Structural Induction works well on functions whose recursive definition follows the definition of the datatype.

Structural Recursion.

Some functions follow other recursive patterns.

> `mergesort` works by splitting its list in two.

We can show that $\phi(l)$ holds for all lists l by showing that:

$$\text{if } \forall\, l'\ \texttt{length}(l') < \texttt{length}(l) \rightarrow \phi(l')$$
$$\text{then } \phi(l)$$

More generally, *well-founded* induction works on any well-ordered set

Logical reasons for types

Types help us rule out certain programs that don't seem to make sense.

Can we apply a function to itself `f f`?

Sometimes, it might be sensible:
`fn x => x` or `fn x => y`.

But in general it looks very suspicious.

This sort of self-application can lead to inconsistencies in formal logics.

Russell's paradox considers $\{x \mid x \notin x\}$,

To avoid this, Russell introduced a system of types.

Type theory is an *alternative* to set theory as a foundation for mathematics.

There are interesting links between type theory and programming.

Programming reasons for types

- More efficient code, and more effective use of space.
- 'Sanity check' for programs, catching a lot of programming errors before execution.
- Documentation.
- Data hiding and modularization.

At the same time, some programmers find them an irksome restriction. How can we achieve the best balance?

Different typing methods

We can distinguish between

- Strong typing, as in Modula-3, where types must match up exactly.
- Weak typing, as in C, where greater latitude is allowed.

and also between

- Static typing, as in FORTRAN, which happens at compilation
- Dynamic typing, as in LISP, which happens during execution.

ML is statically and strongly typed.

At the same time, *polymorphism* gives great flexibility.

Type Expressions

A type expression is:

- One of the basic types: `unit`, `bool`, `int`, `real`, `string`.
- A type variable α.
- $\sigma \rightarrow \tau$
- $\sigma * \tau$
- σ `list`

Where σ and τ are any type expressions.

Polymorphism

Some functions can have various different types — *polymorphism*. We distinguish between:

- True ('parametric') polymorphism, where all the possible types for an expression are instances of some schematic type.

  ```
  fn x => x
  ```

- Ad hoc polymorphism, or *overloading*, where this isn't so.

  ```
  +
  ```

Overloading is limited to a few special cases.

Except for overloading, the ML type inference system can infer a type for every expression.

Type variables

If an expression has a type involving a variable α then it can also be given any type that results from consistently replacing α by another type expression.

A type σ is more general than τ, $(\sigma \leq \tau)$, when we can substitute types for type variables in σ and get τ. For example:

$$
\begin{array}{rcl}
\alpha & \leq & \texttt{bool} \\
\beta & \leq & \alpha \\
(\alpha \to \alpha) & \leq & (\texttt{int} \to \texttt{int}) \\
(\alpha \to \alpha) & \not\leq & (\texttt{int} \to \texttt{bool}) \\
(\alpha \to \beta) & \leq & (\beta \to \beta) \\
(\alpha \to \beta) & \not\leq & \alpha
\end{array}
$$

Most general types

Every expression in ML that has a type has a most general type.

(Hindley/Milner)

There is an algorithm for finding the most general type of any expression, even if it contains no type information at all.

ML implementations use this algorithm.

Thus, except for overloading, it is never necessary to write down a type.

This makes the ML type system much less onerous than some others.

For every pair of type expressions σ and τ, if they can be unified, there is a *most general unifier*, up to re-naming of type variables.

ML type inference (1)

Example:

```
fn a => (fn f => fn x => f x) (fn x => x)
```

Attach distinct type variables to distinct variables in the expression.

fn (a:α) => (fn (f:β) => fn (x:γ) =>
(f:β) (x:γ)) (fn (x:δ) => (x:δ))

Note:

- previously defined constants get their assigned type.
- distinct instances of polymorphic constants get assigned types with distinct type variables.
- different bindings of the same variable are really different variables.

ML type inference (2)

$f\ x$ can only be well-typed if $f : \sigma \to \tau$ and $x : \sigma$ for some σ and τ.

$(f\ x) : \tau$.

`fn (x:`β`) => E:`γ has type $\beta \to \gamma$.

Using these facts, we can find relations among the type variables.

Essentially, we get a series of simultaneous equations, and use them to eliminate some unknowns.

The remaining unknowns, if any, parametrize the final polymorphic type.

If the types can't be matched up, or some type variable has to be equal to some composite type containing itself, then typechecking fails.

It is a form of term *unification*.

Unification

To solve a list of equations:

$$t_1 = s_1, \ldots, t_n = s_n$$

- if t_1 is a variable, replace all occurrences of it in the rest of the list by s_1;
- otherwise, if s_1 is a variable, replace all occurrences of it in the rest of the list by t_1;
- otherwise, if they are distinct constants, then fail;
- otherwise, if they have distinct head constructors ($\rightarrow, *, \texttt{list}$) then fail;
- otherwise, add to the list the list of equations of their corresponding parts;

Repeat until all equations are processed.

ML type inference (3)

For `(f:`β`)` `(x:`γ`)` to be well-typed,

For some ϵ
$\beta = \gamma \rightarrow \epsilon$.

`(fn f => fn x => f x):`$(\gamma \rightarrow \epsilon) \rightarrow (\gamma \rightarrow \epsilon)$

and this is applied to

`(fn x => x):`$\delta \rightarrow \delta$

So $(\gamma \rightarrow \epsilon) = (\delta \rightarrow \delta)$ and
$\gamma = \delta$ and $\epsilon = \delta$

So, the whole expression has type $\alpha \rightarrow (\delta \rightarrow \delta)$.

It doesn't matter how we name the type variables now, so we can call it $\alpha \rightarrow (\beta \rightarrow \beta)$.

Let polymorphism

Let allows us to have local bindings
`let val v = E in E' end`.

This is like `(fn v => E') E`, but typechecking this way leads to a problem,

If `v` is bound to something polymorphic, multiple instances must be allowed to have different types.

```
let val I = fn x => x
 in if I true then I 1 else 0
end;
```

One way to type-check this is by substitution of the bound variable.

There are more efficient ways.

Type preservation

Typechecking ML is static, completed before evaluation.

Evaluation of well-typed expressions cannot give rise to ill-typed expressions.

type preservation.

The main step in evaluation is :

```
(fn x => t[x]) u
  -> t[u]
```

`x` and `u` must have the same types at the outset, so this preserves typability.

The reverse is *not* true,

```
(fn a => fn b => b) (fn x => x x)
```

Pathologies of typechecking

```
fn a => let fun pair x y = fn z => z x y
            val x1 = fn y => pair y y
            val x2 = fn y => x1(x1 y)
            val x3 = fn y => x2(x2 y)
            val x4 = fn y => x3(x3 y)
            val x5 = fn y => x4(x4 y)
         in x5 (fn z => z)
        end;
```

The type of this expression takes about a minute to calculate, and when printed out takes 50,000 lines.

Case Study: Parsing

- Recursive Descent Parsing.
- Parsing functionals that closely correspond to the rules of a grammar.
- Functionals that correspond to concatenation, alternatives, repetition, etc.
- Easy to allow backtracking.
- Contruct syntax trees to which it is easy to attach semantics.
- Similar functionals in other domains.

caveat: The functions are easy to write - the parsing algorithms are the same.

Outline

A lexical analyser converts the string of symbols into a list of tokens.

```
datatype token = Key of string
               | Id of string;
```

`Key` for keywords, and `Id` for identifiers.

A parser is a function of type:

```
type 'a phrase =
     token list -> 'a * token list;
```

The parser removes tokens from the front of the list that match the rule of type `'a`.

Basic Parsers

```
$"string" : string phrase

id : string phrase

empty : 'a list phrase


exception Error of string;

fun id (Id a::toks) = (a, toks)
  | id toks =
      raise Error "Identifier expected";

fun empty toks = ([], toks);
```

Alternatives and Consecutives

```
infix 0 || ;

fun (ph1 || ph2) toks =
  ph1 toks  handle Error _ => ph2 toks;


infix 5 -- ;

fun (ph1 -- ph2) toks =
  let val (x, toks2) = ph1 toks
      val (y, toks3) = ph2 toks2
  in  ((x,y), toks3) end;
```

Transformation and Repetition

```
infix 3 >> ;

fun (ph >> f) toks =
   let val (x, toks2) = ph toks
   in  (f x, toks2) end;


fun repeat ph toks =
    (ph -- repeat ph >> (op::)
     || empty) toks;
```

Propositional Logic 1

Grammar rules:

$$
\begin{aligned}
\mathit{Prop} &= \mathit{Term}\{\texttt{"|"}\ \mathit{Term}\} \\
\mathit{Term} &= \mathit{Factor}\{\texttt{"\&"}\ \mathit{Factor}\} \\
\mathit{Factor} &= \mathit{Id} \mid \texttt{"\~{}"}\ \ \mathit{Factor} \mid \texttt{"("}\ \mathit{Prop}\texttt{")"}
\end{aligned}
$$

```
fun orl(p, []) = p
  | orl(p,(_,q)::l) = orl(Disj(p,q), l);

fun andl(p, []) = p
  | andl(p,(_,q)::l) = andl(Conj(p,q), l);
```

Propositional Logic 2

```
fun prop toks =
  (term -- repeat($"|" -- term)
                  >> orl) toks;

fun term toks =
  (factor -- repeat($"&" -- factor)
                     >> andl) toks;

fun factor toks =
   (   id >> Atom
    || $"~" -- factor
          >> (fn (_,p) => Neg p)
    || $"(" -- prop -- $")"
          >> (fn ((_,p),_) => p)
    ) toks;
```

Backtracking Parser

The parser yields 1 outcome, or an error.

We can obtain a backtracking parser by replacing the exception by a lazy list of outcomes.

```
[]
[(x, toks)]
[(x1, toks1), (x2, toks2),...]
```

Depending on whether the string of tokens is a syntax error, a unique parse, or an ambiguous parse.

`--` and `||` can be modified to combine lazy lists.

The lazy consumption of outcomes leads to backtracking.