Logic and Proof

Computer Science Tripos Part IB
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Contents

Slide 101

Introduction to Logic

Logic concerns *statements* in some *language*

The language can be informal (e.g. English) or *formal*

Some statements are *true*, others *false* or perhaps *meaningless*, . . .

Logic concerns relationships between statements: consistency, entailment, . . .

Logical *proofs* model human reasoning

Slide 102

Statements

Statements are declarative assertions:

> *Black is the colour of my true love's hair.*

They are not greetings, questions, commands, . . . :

> *What is the colour of my true love's hair?*
>
> *I wish my true love had hair.*
>
> *Get a haircut!*

Slide 103

Schematic Statements

The *meta-variables* X, Y, Z, . . . range over ‘real’ objects

Black is the colour of X*’s hair.*

Black is the colour of Y*.*

Z *is the colour of* Y*.*

Schematic statements can express general statements, or questions:

What things are black?

Slide 104

Interpretations and Validity

An *interpretation* maps meta-variables to real objects

The interpretation $\mathsf{Y} \mapsto \mathsf{coal}$ *satisfies* the statement

Black is the colour of Y*.*

but the interpretation $\mathsf{Y} \mapsto \mathsf{strawberries}$ does not!

A statement A is *valid* if all interpretations satisfy A.

Slide 105

Consistency, or Satisfiability

A set S of statements is *consistent* if some interpretation satisfies all elements of S at the same time. Otherwise S is *inconsistent.*

Examples of inconsistent sets:

$$\{X \text{ part of } Y,\ Y \text{ part of } Z,\ X \text{ NOT part of } Z\}$$

$$\{n \text{ is a positive integer},\ n \neq 1,\ n \neq 2,\ \ldots\}$$

satisfiable/unsatisfiable = consistent/inconsistent

Slide 106

Entailment, or Logical Consequence

A set S of statements *entails* A if every interpretation that satisfies all elements of S, also satisfies A. We write $S \models A$.

$$\{X \text{ part of } Y,\ Y \text{ part of } Z\} \models X \text{ part of } Z$$

$$\{n \neq 1,\ n \neq 2,\ \ldots\} \models n \text{ is NOT a positive integer}$$

$S \models A$ if and only if $\{\neg A\} \cup S$ is inconsistent

$\models A$ if and only if A is valid

Slide 107

Inference

Want to check A is valid

Checking all interpretations can be effective — but if there are infinitely many?

Let $\{A_1, \ldots, A_n\} \models B$. If $A_1, \ldots, A_n$ are true then B must be true. Write this as the inference

$$\frac{A_1 \qquad \ldots \qquad A_n}{B}$$

Use inferences to construct finite proofs!

Slide 108

Schematic Inference Rules

$$\frac{X \text{ part of } Y \qquad Y \text{ part of } Z}{X \text{ part of } Z}$$

A valid inference:

$$\frac{\text{spoke part of wheel} \qquad \text{wheel part of bike}}{\text{spoke part of bike}}$$

An inference may be valid even if the premises are false!

$$\frac{\text{cow part of chair} \qquad \text{chair part of ant}}{\text{cow part of ant}}$$

Slide 109

Survey of Formal Logics

propositional logic is traditional *boolean algebra*.

first-order logic can say *for all* and *there exists*.

higher-order logic reasons about sets and functions. It has been applied to hardware verification.

modal/temporal logics reason about what *must*, or *may*, happen.

type theories support *constructive* mathematics.

Slide 201

Syntax of Propositional Logic

$P, Q, R, \ldots$	propositional letter
$\mathbf{t}$	true
$\mathbf{f}$	false
$\neg A$	not A
$A \land B$	A and B
$A \lor B$	A or B
$A \to B$	if A then B
$A \leftrightarrow B$	A if and only if B

Slide 202

Semantics of Propositional Logic

$\neg$, $\land$, $\lor$, $\to$ and $\leftrightarrow$ are *truth-functional*: functions of their operands

A	B	$\neg A$	$A \land B$	$A \lor B$	$A \to B$	$A \leftrightarrow B$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$

Slide 203

Interpretations of Propositional Logic

An *interpretation* is a function from the propositional letters to $\{\mathbf{t}, \mathbf{f}\}$.

Interpretation I *satisfies* a formula A if the formula evaluates to $\mathbf{t}$.

Write $\models_I A$

A is *valid* (a *tautology*) if every interpretation satisfies A

Write $\models A$

S is *satisfiable* if some interpretation satisfies every formula in S

Slide 204

Implication, Entailment, Equivalence

$A \to B$ means simply $\neg A \vee B$

$A \models B$ means if $\models_I A$ then $\models_I B$ for every interpretation I

$A \models B$ if and only if $\models A \to B$

Equivalence

$A \simeq B$ means $A \models B$ and $B \models A$

$A \simeq B$ if and only if $\models A \leftrightarrow B$

Slide 205

Equivalences

$$A \land A \simeq A$$
$$A \land B \simeq B \land A$$
$$(A \land B) \land C \simeq A \land (B \land C)$$
$$A \lor (B \land C) \simeq (A \lor B) \land (A \lor C)$$
$$A \land \mathbf{f} \simeq \mathbf{f}$$
$$A \land \mathbf{t} \simeq A$$
$$A \land \neg A \simeq \mathbf{f}$$

Dual versions: exchange $\land$, $\lor$ and $\mathbf{t}$, $\mathbf{f}$ in any equivalence

Slide 206

Negation Normal Form

1. Get rid of $\leftrightarrow$ and $\rightarrow$, leaving just $\land$, $\lor$, $\neg$:

$$A \leftrightarrow B \simeq (A \rightarrow B) \land (B \rightarrow A)$$
$$A \rightarrow B \simeq \neg A \lor B$$

2. Push negations in, using de Morgan's laws:

$$\neg\neg A \simeq A$$
$$\neg(A \land B) \simeq \neg A \lor \neg B$$
$$\neg(A \lor B) \simeq \neg A \land \neg B$$

Slide 207

From NNF to Conjunctive Normal Form

3. Push disjunctions in, using distributive laws:

$$A \lor (B \land C) \simeq (A \lor B) \land (A \lor C)$$
$$(B \land C) \lor A \simeq (B \lor A) \land (C \lor A)$$

4. Simplify:

- Delete any disjunction containing P and $\neg P$
- Delete any disjunction that includes another
- Replace $(P \lor A) \land (\neg P \lor A)$ by A

Slide 208

Converting a Non-Tautology to CNF

$$P \lor Q \to Q \lor R$$

1. Elim $\to$: $\neg(P \lor Q) \lor (Q \lor R)$
2. Push $\neg$ in: $(\neg P \land \neg Q) \lor (Q \lor R)$
3. Push $\lor$ in: $(\neg P \lor Q \lor R) \land (\neg Q \lor Q \lor R)$
4. Simplify: $\neg P \lor Q \lor R$

Not a tautology: try $P \mapsto \mathbf{t}$, $Q \mapsto \mathbf{f}$, $R \mapsto \mathbf{f}$

Slide 209

Tautology checking using CNF

$$((P \to Q) \to P) \to P$$

1. Elim $\to$: $\neg[\neg(\neg P \vee Q) \vee P] \vee P$
2. Push $\neg$ in: $[\neg\neg(\neg P \vee Q) \wedge \neg P] \vee P$

   $[(\neg P \vee Q) \wedge \neg P] \vee P$
3. Push $\vee$ in: $(\neg P \vee Q \vee P) \wedge (\neg P \vee P)$
4. Simplify: $\mathbf{t} \wedge \mathbf{t}$

   $\mathbf{t}$ *It's a tautology!*

Slide 301

A Simple Proof System

Axiom Schemes

K	$A \to (B \to A)$
S	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
DN	$\neg\neg A \to A$

Inference Rule: Modus Ponens

$$\frac{A \to B \qquad A}{B}$$

Slide 302

A Simple (?) Proof of $A \to A$

$(A \to ((D \to A) \to A)) \to$		(1)
$((A \to (D \to A)) \to (A \to A))$	by S	
$A \to ((D \to A) \to A)$	by K	(2)
$(A \to (D \to A)) \to (A \to A)$	by MP, (1), (2)	(3)
$A \to (D \to A)$	by K	(4)
$A \to A$	by MP, (3), (4)	(5)

Slide 303

Some Facts about Deducibility

A is *deducible from* the set S of if there is a finite proof of A starting from elements of S. Write $S \vdash A$.

Soundness Theorem. If $S \vdash A$ then $S \models A$.

Completeness Theorem. If $S \models A$ then $S \vdash A$.

Deduction Theorem. If $S \cup \{A\} \vdash B$ then $S \vdash A \to B$.

Slide 304

Gentzen's Natural Deduction Systems

A varying context of *assumptions*

Each logical connective defined *independently*

Introduction rule for $\wedge$: how to deduce $A \wedge B$

$$\frac{A \qquad B}{A \wedge B}$$

Elimination rules for $\wedge$: what to deduce *from* $A \wedge B$

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

Slide 305

The Sequent Calculus

Sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ means,

$$\text{if } A_1 \land \ldots \land A_m \text{ then } B_1 \lor \ldots \lor B_n$$

$A_1, \ldots, A_m$ are *assumptions*; $B_1, \ldots, B_n$ are *goals*

Γ and Δ are *sets* in $\Gamma \Rightarrow \Delta$

$A, \Gamma \Rightarrow A, \Delta$ is trivially true (*basic sequent*)

Slide 306

Sequent Calculus Rules

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \;(\mathrm{cut})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \;(\neg l) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \;(\neg r)$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \land B, \Gamma \Rightarrow \Delta} \;(\land l) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \land B} \;(\land r)$$

Slide 307

More Sequent Calculus Rules

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee r)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ (\to l) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to r)$$

Slide 308

Easy Sequent Calculus Proofs

$$\dfrac{\dfrac{\overline{A, B \Rightarrow A}}{A \wedge B \Rightarrow A}\ (\wedge l)}{\Rightarrow A \wedge B \to A}\ (\to r)$$

$$\dfrac{\dfrac{\dfrac{\overline{A, B \Rightarrow B, A}}{A \Rightarrow B, B \to A}\ (\to r)}{\Rightarrow A \to B,\ B \to A}\ (\to r)}{\Rightarrow (A \to B) \vee (B \to A)}\ (\vee r)$$

Slide 309

Part of a Distributive Law

$$\dfrac{\dfrac{}{A \Rightarrow A, B} \qquad \dfrac{\dfrac{}{B, C \Rightarrow A, B}}{B \wedge C \Rightarrow A, B}\,(\wedge l)}{\dfrac{A \vee (B \wedge C) \Rightarrow A, B}{A \vee (B \wedge C) \Rightarrow A \vee B}\,(\vee r)}\,(\vee l)$$

$$\dfrac{A \vee (B \wedge C) \Rightarrow A \vee B \qquad \text{similar}}{A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)}\,(\wedge r)$$

Second subtree proves $A \vee (B \wedge C) \Rightarrow A \vee C$ similarly

Slide 310

A Failed Proof

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B, C \qquad \dfrac{}{B \Rightarrow B, C}}{A \vee B \Rightarrow B, C}\,(\vee l)}{A \vee B \Rightarrow B \vee C}\,(\vee r)}{\Rightarrow A \vee B \to B \vee C}\,(\to r)$$

$A \mapsto \mathbf{t},\ B \mapsto \mathbf{f},\ C \mapsto \mathbf{f}$ falsifies unproved sequent!

Slide 401

Ordered Binary Decision Diagrams

Canonical form: essentially decision trees with sharing

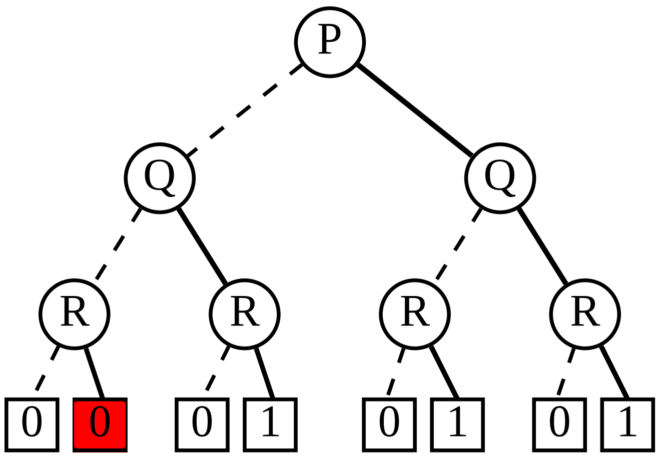
- *ordered* propositional symbols ('variables')
- *sharing* of identical subtrees
- *hashing* and other optimisations

Detects if a formula is tautologous ($\mathbf{t}$) or inconsistent ($\mathbf{f}$)

A **FAST** way of verifying digital circuits, . . .

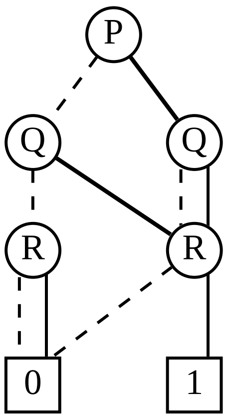
Slide 402

Decision Diagram for $(P \lor Q) \land R$

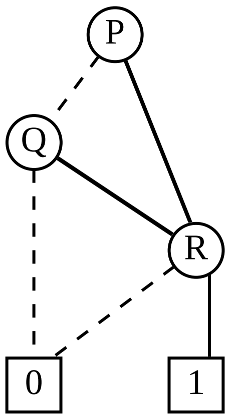


Slide 403

Converting a Decision Diagram to an OBDD



No duplicates



No redundant tests

Slide 404

Building OBDDs Efficiently

Do not construct full tree! (see Bryant, §3.1)

Do not expand $\to$, $\leftrightarrow$, $\oplus$ (exclusive OR) to other connectives

Treat $\neg Z$ as $Z \to \mathbf{f}$ or $Z \oplus \mathbf{t}$

Recursively convert operands

Combine operand OBDDs — respecting ordering and sharing

Delete test if it proves to be redundant

Slide 405

Canonical Form Algorithm

To do $Z \wedge Z'$, where Z and Z' are already canonical:

Trivial if either is $\mathbf{t}$ *or* $\mathbf{f}$. *Treat* $\vee$, $\to$, $\leftrightarrow$ *similarly!*

Let $Z = \mathbf{if}(P, X, Y)$ and $Z' = \mathbf{if}(P', X', Y')$

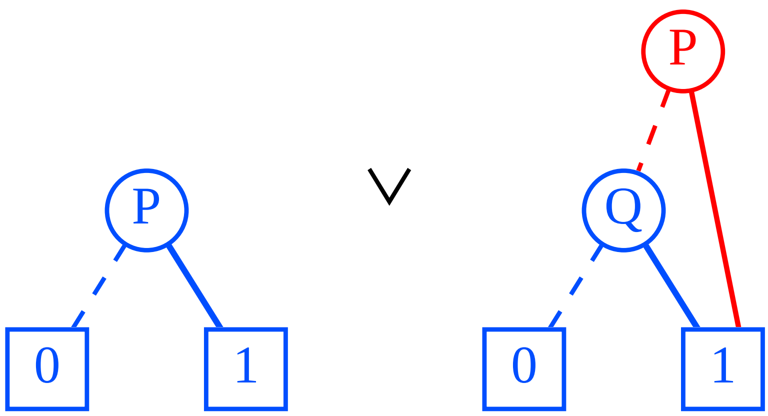
If $P = P'$ then recursively do $\mathbf{if}(P,\ X \wedge X',\ Y \wedge Y')$

If $P < P'$ then recursively do $\mathbf{if}(P,\ X \wedge Z',\ Y \wedge Z')$

If $P > P'$ then recursively do $\mathbf{if}(P',\ Z \wedge X',\ Z \wedge Y')$

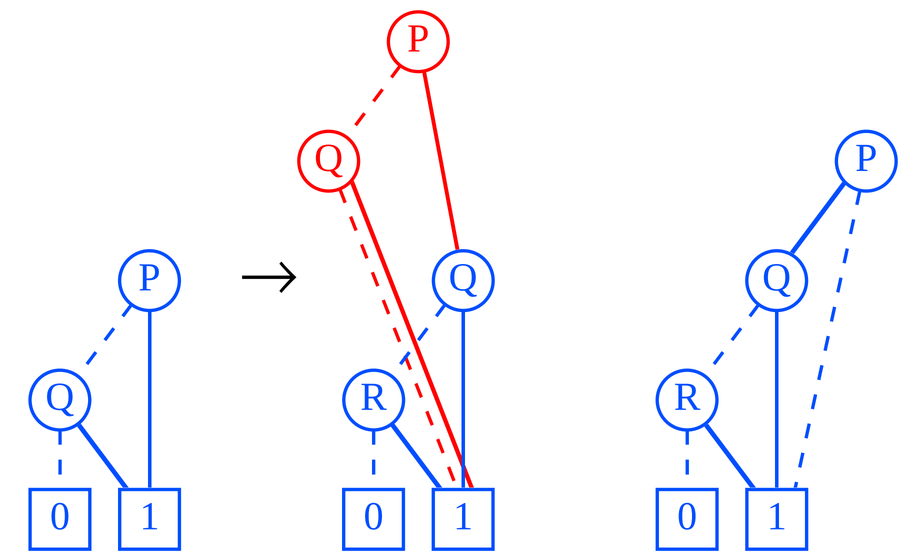
Slide 406

Canonical Form of $P \vee Q$



Slide 407

Canonical Form of $P \vee Q \to Q \vee R$



Slide 408

Optimisations Based On Hash Tables

Never build the same OBDD twice: share pointers

- Pointer identity: $X = Y$ whenever $X \leftrightarrow Y$
- Fast removal of redundant tests by $\textbf{if}(P, X, X) \simeq X$
- Fast processing of $X \wedge X$, $X \vee X$, $X \to X$, ...

Never process $X \wedge Y$ twice; keep table of canonical forms

Slide 409

Final Observations

The variable ordering is crucial. Consider

$$(P_1 \wedge Q_1) \vee \cdots \vee (P_n \wedge Q_n)$$

A good ordering is $P_1 < Q_1 < \cdots < P_n < Q_n$

A dreadful ordering is $P_1 < \cdots < P_n < Q_1 < \cdots < Q_n$

Many digital circuits have small OBDDs *(not multiplication!)*

OBDDs can solve problems in hundreds of variables

General case remains intractable!

Slide 501

Outline of First-Order Logic

Reasons about *functions* and *relations* over a set of *individuals*

$$\frac{\mathsf{father}(\mathsf{father}(x)) = \mathsf{father}(\mathsf{father}(y))}{\mathsf{cousin}(x, y)}$$

Reasons about *all* and *some* individuals:

$$\frac{\text{All men are mortal} \qquad \text{Socrates is a man}}{\text{Socrates is mortal}}$$

Does not reason about *all functions* or *all relations*, . . .

Slide 502

Function Symbols; Terms

Each *function symbol* stands for an n-place function

A *constant symbol* is a 0-place function symbol

A *variable* ranges over all individuals

A *term* is a variable, constant or has the form

$$f(t_1, \ldots, t_n)$$

where f is an n-place function symbol and t_1, . . . , t_n are terms

We choose the language, adopting any desired function symbols

Slide 503

Relation Symbols; Formulae

Each *relation symbol* stands for an n-place relation

Equality is the 2-place relation symbol $=$

An *atomic formula* has the form

$$R(t_1, \ldots, t_n)$$

where R is an n-place relation symbol and $t_1, \ldots, t_n$ are terms

A *formula* is built up from atomic formulæ using $\neg$, $\land$, $\lor$, ...

(Later we add *quantifiers*)

Slide 504

Power of Quantifier-Free FOL

Very expressive, given strong induction rules

Prove equivalence of mathematical functions:

$$\begin{aligned} p(z,0) &= 1 \\ p(z,n+1) &= p(z,n) \times z \end{aligned} \qquad \begin{aligned} q(z,1) &= z \\ q(z,2 \times n) &= q(z \times z, n) \\ q(z,2 \times n+1) &= q(z \times z, n) \times z \end{aligned}$$

Boyer/Moore Theorem Prover: checked Gödel's Theorem, ...

Many systems based on *equational reasoning*

Slide 505

Universal and Existential Quantifiers

$\forall x\, A$	for all x, A holds
$\exists x\, A$	there exists x such that A holds

Syntactic variations:

$\forall xyz\, A$	abbreviates $\forall x\, \forall y\, \forall z\, A$
$\forall z\, .\, A \wedge B$	is an alternative to $\forall z\, (A \wedge B)$

The variable x is *bound* in $\forall x\, A$; compare with $\int f(x)\, dx$

Slide 506

Expressiveness of Quantifiers

All men are mortal:

$$\forall x\, (\mathsf{man}(x) \to \mathsf{mortal}(x))$$

All mothers are female:

$$\forall x\, \mathsf{female}(\mathsf{mother}(x))$$

There exists a unique x *such that* A, written $\exists! x\, A$

$$\exists x\, [A(x) \wedge \forall y\, (A(y) \to y = x)]$$

Slide 507

How do we interpret $\mathsf{mortal}(\mathsf{Socrates})$?

Interpretation $\mathcal{I} = (D, I)$ of our first-order language

D is a non-empty *universe*

I maps symbols to 'real' functions, relations

c a constant symbol	$I[c] \in D$
f an n-place function symbol	$I[f] \in D^n \to D$
P an n-place relation symbol	$I[P] \subseteq D^n$

Slide 508

How do we interpret $\mathsf{cousin}(\mathsf{Charles}, y)$?

A *valuation* supplies the values of free variables

It is a function $V : \mathrm{variables} \to D$

$\mathcal{I}_V[t]$ extends V to a term t by the obvious recursion:

$$\mathcal{I}_V[x] \stackrel{\mathrm{def}}{=} V(x) \qquad \text{if } x \text{ is a variable}$$

$$\mathcal{I}_V[c] \stackrel{\mathrm{def}}{=} I[c]$$

$$\mathcal{I}_V[f(t_1, \ldots, t_n)] \stackrel{\mathrm{def}}{=} I[f](\mathcal{I}_V[t_1], \ldots, \mathcal{I}_V[t_n])$$

Slide 509

The Meaning of Truth — in FOL

For interpretation $\mathcal{I}$ and valuation V

$\models_{\mathcal{I},V} P(t)$	if $I[P](\mathcal{I}_V[t])$ holds
$\models_{\mathcal{I},V} t = u$	if $\mathcal{I}_V[t]$ equals $\mathcal{I}_V[u]$
$\models_{\mathcal{I},V} A \land B$	if $\models_{\mathcal{I},V} A$ and $\models_{\mathcal{I},V} B$
$\models_{\mathcal{I},V} \exists x\, A$	if $\models_{\mathcal{I},V\{m/x\}} A$ holds for some $m \in D$
$\models_{\mathcal{I}} A$	if $\models_{\mathcal{I},V} A$ holds for all V

A is *satisfiable* if $\models_{\mathcal{I}} A$ for some $\mathcal{I}$

Slide 601

Free v Bound Variables

All occurrences of x in $\forall x\, A$ and $\exists x\, A$ are *bound*

An occurrence of x is *free* if it is not bound:

$$\forall x\, \exists y\, R(x, y, f(x, z))$$

May *rename* bound variables:

$$\forall w\, \exists y'\, R(w, y', f(w, z))$$

Slide 602

Substitution for Free Variables

$A[t/x]$ means 'substitute t for x in A':

$(B \land C)[t/x]$ is $B[t/x] \land C[t/x]$

$(\forall x\, B)[t/x]$ is $\forall x\, B$

$(\forall y\, B)[t/x]$ is $\forall y\, B[t/x] \qquad (x \neq y)$

$(P(u))[t/x]$ is $P(u[t/x])$

No variable in t may be bound in A!

$(\forall y\, x = y)[y/x]$ is not $\forall y\, y = y$!

Slide 603

Some Equivalences for Quantifiers

$$\neg(\forall x\, A) \simeq \exists x\, \neg A$$

$$(\forall x\, A) \land B \simeq \forall x\, (A \land B)$$

$$(\forall x\, A) \lor B \simeq \forall x\, (A \lor B)$$

$$(\forall x\, A) \land (\forall x\, B) \simeq \forall x\, (A \land B)$$

$$(\forall x\, A) \to B \simeq \exists x\, (A \to B)$$

$$\forall x\, A \simeq \forall x\, A \land A[t/x]$$

Dual versions: exchange $\forall$, $\exists$ and $\land$, $\lor$

Slide 604

Reasoning by Equivalences

$$\begin{aligned}\exists x\, (x = a \land P(x)) &\simeq \exists x\, (x = a \land P(a)) \\ &\simeq \exists x\, (x = a) \land P(a) \\ &\simeq P(a)\end{aligned}$$

$$\begin{aligned}&\exists z\, (P(z) \to P(a) \land P(b)) \\ &\qquad \simeq \forall z\, P(z) \to P(a) \land P(b) \\ &\qquad \simeq \forall z\, P(z) \land P(a) \land P(b) \to P(a) \land P(b) \\ &\qquad \simeq \mathbf{t}\end{aligned}$$

Slide 605

Sequent Calculus Rules for $\forall$

$$\frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall x\, A, \Gamma \Rightarrow \Delta}\ (\forall l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x\, A}\ (\forall r)$$

Rule $(\forall l)$ can create many instances of $\forall x\, A$

Rule $(\forall r)$ holds *provided* x is not free in the conclusion!

Not allowed to prove

$$\frac{\overline{P(y) \Rightarrow P(y)}}{P(y) \Rightarrow \forall y\, P(y)}\ (\forall r)$$

Slide 606

Examples of the $\forall$ Rules

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{A \Rightarrow B, A} \qquad \overline{A, B \Rightarrow B}}{A,\ A \to B \Rightarrow B}\ (\to l)}{A,\ \forall x\,(A \to B) \Rightarrow B}\ (\forall l)}{A,\ \forall x\,(A \to B) \Rightarrow \forall x\, B}\ (\forall r)}{\forall x\,(A \to B) \Rightarrow A \to \forall x\, B}\ (\to r)$$

$$\cfrac{\cfrac{\overline{P(f(y)) \Rightarrow P(f(y))}}{\forall x\, P(x) \Rightarrow P(f(y))}\ (\forall l)}{\forall x\, P(x) \Rightarrow \forall y\, P(f(y))}\ (\forall r)$$

x must not be free in A !

Slide 607

Sequent Calculus Rules for $\exists$

$$\frac{A, \Gamma \Rightarrow \Delta}{\exists x\, A, \Gamma \Rightarrow \Delta}\ (\exists l) \qquad \frac{\Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x\, A}\ (\exists r)$$

Rule $(\exists l)$ holds *provided* x is not free in the conclusion!

Rule $(\exists r)$ can create many instances of $\exists x\, A$

Say, to prove

$$\exists z\, (P(z) \to P(a) \land P(b))$$

Slide 608

Part of the $\exists$ Distributive Law

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow A, B}}{A \Rightarrow A \lor B}\ (\lor r)}{A \Rightarrow \exists x\, (A \lor B)}\ (\exists r)}{\exists x\, A \Rightarrow \exists x\, (A \lor B)}\ (\exists l) \qquad \dfrac{\text{similar}}{\exists x\, B \Rightarrow \exists x\, (A \lor B)}\ (\exists l)}{\exists x\, A \lor \exists x\, B \Rightarrow \exists x\, (A \lor B)}\ (\lor l)$$

Second subtree proves $\exists x\, B \Rightarrow \exists x\, (A \lor B)$ similarly

Slide 609

A Failed Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{A, B[y/x] \Rightarrow A \land B}{A, B[y/x] \Rightarrow \exists x\,(A \land B)}\,(\exists r)}{A, \exists x\,B \Rightarrow \exists x\,(A \land B)}\,(\exists l)}{\exists x\,A, \exists x\,B \Rightarrow \exists x\,(A \land B)}\,(\exists l)}{\exists x\,A \land \exists x\,B \Rightarrow \exists x\,(A \land B)}\,(\land l)
$$

Cannot use $(\exists l)$ twice with the same x

We can easily falsify the topmost sequent

Slide 701

Clause Form

Clause: a disjunction of *literals*

$$\neg K_1 \vee \cdots \vee \neg K_m \vee L_1 \vee \cdots \vee L_n$$

Set notation: $\{\neg K_1, \ldots, \neg K_m, L_1, \ldots, L_n\}$

Kowalski notation: $K_1, \cdots, K_m \to L_1, \cdots, L_n$

$L_1, \cdots, L_n \leftarrow K_1, \cdots, K_m$

Empty clause $\square$

Empty clause means contradiction!

Slide 702

Outline of Clause Form Methods

To prove A, obtain a contradiction from $\neg A$:

1. Translate $\neg A$ into CNF as $A_1 \wedge \cdots \wedge A_m$
2. This is the set of clauses $A_1, \ldots, A_m$
3. Transform the clause set, preserving consistency

Empty *clause* refutes $\neg A$

Empty *clause set* means $\neg A$ is satisfiable

Slide 703

The Davis-Putnam Decision Procedure

1. Delete tautological clauses: $\{P, \neg P, \ldots\}$
2. For each unit clause $\{L\}$,
   - delete all clauses containing L
   - delete $\neg L$ from all clauses
3. Delete all clauses containing *pure literals*
4. Perform a *case split* on some literal

Slide 704

Davis-Putnam on a Non-Tautology

Consider $P \vee Q \to Q \vee R$

Clauses are $\{P, Q\} \quad \{\neg Q\} \quad \{\neg R\}$

$\{P, Q\}$	$\{\neg Q\}$	$\{\neg R\}$	initial clauses
$\{P\}$		$\{\neg R\}$	unit $\neg Q$
		$\{\neg R\}$	unit P (also pure)
			unit $\neg R$ (also pure)

Clauses satisfiable by $P \mapsto \mathbf{t},\ Q \mapsto \mathbf{f},\ R \mapsto \mathbf{f}$

Slide 705

Example of a Case Split on P

$\{\neg Q, R\}$ $\{\neg R, P\}$ $\{\neg R, Q\}$ $\{\neg P, Q, R\}$ $\{P, Q\}$ $\{\neg P, \neg Q\}$

$\{\neg Q, R\}$	$\{\neg R, Q\}$	$\{Q, R\}$	$\{\neg Q\}$	if P is true
	$\{\neg R\}$	$\{R\}$		unit $\neg Q$
	$\square$			unit R
$\{\neg Q, R\}$	$\{\neg R\}$	$\{\neg R, Q\}$	$\{Q\}$	if P is false
$\{\neg Q\}$			$\{Q\}$	unit $\neg R$
			$\square$	unit $\neg Q$

Slide 706

The Resolution Rule

From $B \vee A$ and $\neg B \vee C$ infer $A \vee C$

In set notation,

$$\frac{\{B, A_1, \ldots, A_m\} \qquad \{\neg B, C_1, \ldots, C_n\}}{\{A_1, \ldots, A_m, C_1, \ldots, C_n\}}$$

Some special cases:

$$\frac{\{B\} \qquad \{\neg B, C_1, \ldots, C_n\}}{\{C_1, \ldots, C_n\}} \qquad\qquad \frac{\{B\} \qquad \{\neg B\}}{\square}$$

Slide 707

Simple Example: Proving $P \land Q \to Q \land P$

Hint: use $\neg(A \to B) \simeq A \land \neg B$

1. Negate!	$\neg[P \land Q \to Q \land P]$
2. Push $\neg$ in:	$(P \land Q) \land \neg(Q \land P)$
	$(P \land Q) \land (\neg Q \lor \neg P)$
Clauses:	$\{P\} \quad \{Q\} \quad \{\neg Q, \neg P\}$

Resolve $\{P\}$ and $\{\neg Q, \neg P\}$ getting $\{\neg Q\}$

Resolve $\{Q\}$ and $\{\neg Q\}$ getting $\square$

Slide 708

Another Example

Refute $\neg[(P \lor Q) \land (P \lor R) \to P \lor (Q \land R)]$

From $(P \lor Q) \land (P \lor R)$, get clauses $\{P, Q\}$ and $\{P, R\}$

From $\neg\ [P \lor (Q \land R)]$ get clauses $\{\neg P\}$ and $\{\neg Q, \neg R\}$

Resolve $\{\neg P\}$ and $\{P, Q\}$ getting $\{Q\}$

Resolve $\{\neg P\}$ and $\{P, R\}$ getting $\{R\}$

Resolve $\{Q\}$ and $\{\neg Q, \neg R\}$ getting $\{\neg R\}$

Resolve $\{R\}$ and $\{\neg R\}$ getting $\square$

Slide 709

Refinements of Resolution

Preprocessing:	removing tautologies, symmetries . . .
Set of Support:	working from the goal
Weighting:	priority to the smallest clauses
Subsumption:	deleting redundant clauses
Hyper-resolution:	avoiding intermediate clauses
Indexing:	data structures for speed

Slide 801

Reducing FOL to Propositional Logic

Prenex:	Move quantifiers to the front
Skolemize:	Remove quantifiers, preserving consistency
Herbrand models:	Reduce the class of interpretations
Herbrand's Thm:	Contradictions have *finite*, *ground* proofs
Unification:	Automatically find the right instantiations

Finally, combine unification with *resolution*

Slide 802

Prenex Normal Form

Convert to Negation Normal Form using additionally

$$\neg(\forall x\, A) \simeq \exists x\, \neg A$$
$$\neg(\exists x\, A) \simeq \forall x\, \neg A$$

Then move quantifiers to the front using

$$(\forall x\, A) \land B \simeq \forall x\, (A \land B)$$
$$(\forall x\, A) \lor B \simeq \forall x\, (A \lor B)$$

and the similar rules for $\exists$

Slide 803

Skolemization

Take a formula of the form

$$\forall x_1\, \forall x_2\, \cdots \forall x_k\, \exists y\, A$$

Choose a new k-place function symbol, say f

Delete $\exists y$ and replace y by $f(x_1, x_2, \ldots, x_k)$. We get

$$\forall x_1\, \forall x_2\, \cdots \forall x_k\, A[f(x_1, x_2, \ldots, x_k)/y]$$

Repeat until no $\exists$ quantifiers remain

Slide 804

Example of Conversion to Clauses

For proving $\exists x\, [P(x) \to \forall y\, P(y)]$

$\neg\, [\exists x\, [P(x) \to \forall y\, P(y)]]$	negated goal
$\forall x\, [P(x) \land \exists y\, \neg P(y)]$	conversion to NNF
$\forall x\, \exists y\, [P(x) \land \neg P(y)]$	pulling $\exists$ out
$\forall x\, [P(x) \land \neg P(f(x))]$	Skolem term $f(x)$
$\{P(x)\} \quad \{\neg P(f(x))\}$	Final clauses

Slide 805

Correctness of Skolemization

The formula $\forall x\, \exists y\, A$ is consistent

$\iff$ it holds in some interpretation $\mathcal{I} = (D, I)$

$\iff$ for all $x \in D$ there is some $y \in D$ such that A holds

$\iff$ some function $\hat{f}$ in $D \to D$ yields suitable values of y

$\iff$ $A[f(x)/y]$ holds in some $\mathcal{I}'$ extending $\mathcal{I}$ so that f denotes $\hat{f}$

$\iff$ the formula $\forall x\, A[f(x)/y]$ is consistent.

Slide 806

Herbrand Interpretations for a set of clauses S

$$H_0 \stackrel{\text{def}}{=} \text{the set of constants in } S$$

$$H_{i+1} \stackrel{\text{def}}{=} H_i \cup \{f(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in H_i \text{ and } f \text{ is an } n\text{-place function symbol in } S\}$$

$$H \stackrel{\text{def}}{=} \bigcup_{i \geq 0} H_i \qquad \textit{Herbrand Universe}$$

$$HB \stackrel{\text{def}}{=} \{P(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in H \text{ and } P \text{ is an } n\text{-place predicate symbol in } S\}$$

Slide 807

Example of an Herbrand Model

$$\left.\begin{array}{r}\neg even(1)\\ even(2)\\ even(X\cdot Y)\leftarrow even(X), even(Y)\end{array}\right\}\text{ clauses}$$

$$\begin{aligned} H &= \{1, 2, 1\cdot 1, 1\cdot 2, 2\cdot 1, 2\cdot 2, 1\cdot(1\cdot 1), \ldots\}\\ HB &= \{even(1), even(2), even(1\cdot 1), even(1\cdot 2), \ldots\}\\ I[even] &= \{even(2), even(1\cdot 2), even(2\cdot 1), even(2\cdot 2), \ldots\}\end{aligned}$$

(for model where $\cdot$ means product; could instead use sum!)

Slide 808

A Key Fact about Herbrand Interpretations

Let S be a set of clauses.

S is unsatisfiable $\iff$ no Herbrand interpretation satisfies S

- Holds because some Herbrand model mimicks every ‘real’ model
- We must consider only a small class of models
- Herbrand models are syntactic, easily processed by computer

Slide 809

Herbrand's Theorem

Let S be a set of clauses.

S *is unsatisfiable* $\iff$ *there is a* finite *unsatisfiable set* S' *of* ground instances *of clauses of* S.

- **Finite**: we can compute it
- **Instance**: result of substituting for variables
- **Ground**: and no variables remain: it's propositional!

Slide 901

Unification

Finding a common instance of two terms

- Logic programming (Prolog)
- Polymorphic type-checking (ML)
- Constraint satisfaction problems
- Resolution theorem proving for FOL
- Many other theorem proving methods

Slide 902

Substitutions

A finite set of *replacements*

$$\theta = [t_1/x_1, \ldots, t_k/x_k]$$

where $x_1, \ldots, x_k$ are distinct variables and $t_i \neq x_i$

$$f(t, u)\theta = f(t\theta, u\theta) \qquad \text{(terms)}$$

$$P(t, u)\theta = P(t\theta, u\theta) \qquad \text{(literals)}$$

$$\{L_1, \ldots, L_m\}\theta = \{L_1\theta, \ldots, L_m\theta\} \qquad \text{(clauses)}$$

Slide 903

Composing Substitutions

Composition of ϕ and θ, written $\phi \circ \theta$, satisfies for all terms t

$$t(\phi \circ \theta) = (t\phi)\theta$$

It is defined by (for all relevant x)

$$\phi \circ \theta \stackrel{\text{def}}{=} [\,(x\phi)\theta \,/\, x, \ldots\,]$$

Consequences include $\theta \circ [] = \theta$, and *associativity*:

$$(\phi \circ \theta) \circ \sigma = \phi \circ (\theta \circ \sigma)$$

Slide 904

Most General Unifiers

θ is a *unifier* of terms t and u if $t\theta = u\theta$

θ is *more general* than ϕ if $\phi = \theta \circ \sigma$

θ is *most general* if it is more general than every other unifier

If θ unifies t and u then so does $\theta \circ \sigma$:

$$t(\theta \circ \sigma) = t\theta\sigma = u\theta\sigma = u(\theta \circ \sigma)$$

A most general unifier of $f(a, x)$ and $f(y, g(z))$ is $[a/y, g(z)/x]$

The common instance is $f(a, g(z))$

Slide 905

Algorithm for Unifying Two Terms

Represent terms by *binary trees*

Each term is a *Variable* $x, y \ldots$, *Constant* $a, b \ldots$, or *Pair* (t, t')

Constants do not unify with different Constants

Constants do not unify with Pairs

Variable x and term t: unifier is $[t/x]$ — **unless** x occurs in t

Cannot unify $f(x)$ with x!

Slide 906

Unifying Two Pairs

$\theta \circ \theta'$ unifies (t, t') with (u, u')

if θ unifies t with u and θ' unifies $t'\theta$ with $u'\theta$

$$\begin{aligned}
(t, t')(\theta \circ \theta') &= (t, t')\theta\theta' \\
&= (t\theta\theta', t'\theta\theta') \\
&= (u\theta\theta', u'\theta\theta') \\
&= (u, u')\theta\theta' \\
&= (u, u')(\theta \circ \theta')
\end{aligned}$$

Slide 907

Examples of Unification

$f(x, b)$	$f(x, x)$	$f(x, x)$	$j(x, x, z)$
$f(a, y)$	$f(a, b)$	$f(y, g(y))$	$j(w, a, h(w))$
$f(a, b)$	?	?	$j(a, a, h(a))$
$[a/x, b/y]$	**FAIL**	**FAIL**	$[a/w, a/x, h(a)/z]$

We always get a **most general** unifier

Slide 908

Theorem-Proving Examples

$$(\exists y\, \forall x\, R(x, y)) \to (\forall x\, \exists y\, R(x, y))$$

Clauses after negation are $\{R(x, a)\}$ and $\{\neg R(b, y)\}$

$R(x, a)$ and $R(b, y)$ have unifier $[b/x, a/y]$: *contradiction!*

$$(\forall x\, \exists y\, R(x, y)) \to (\exists y\, \forall x\, R(x, y))$$

Clauses after negation are $\{R(x, f(x))\}$ and $\{\neg R(g(y), y)\}$

$R(x, f(x))$ and $R(g(y), y)$ are not unifiable: *occurs check*

Formula is not a theorem!

Slide 909

Variations on Unification

Efficient unification algorithms: near-linear time

Indexing & Discrimination networks: fast retrieval of a unifiable term

Order-sorted unification: type-checking in Haskell

Associative/commutative operators: problems in group theory

Higher-order unification: support λ-calculus

Boolean unification: reasoning about sets

Slide 1001

Binary Resolution

$$\frac{\{B, A_1, \ldots, A_m\} \qquad \{\neg D, C_1, \ldots, C_n\}}{\{A_1, \ldots, A_m, C_1, \ldots, C_n\}\sigma} \qquad \textit{provided } B\sigma = D\sigma$$

First *rename variables apart* in the clauses! — say, to resolve

$$\{P(x)\} \quad \text{and} \quad \{\neg P(g(x))\}$$

Always use a *most general* unifier (MGU)

Soundness? Same argument as for the propositional version

Slide 1002

Factorisation

Collapsing similar literals *in one clause*:

$$\frac{\{B_1, \ldots, B_k, A_1, \ldots, A_m\}}{\{B_1, A_1, \ldots, A_m\}\sigma} \qquad \textit{provided } B_1\sigma = \cdots = B_k\sigma$$

Normally combined with resolution

Prove $\forall x \, \exists y \, \neg(P(y,x) \leftrightarrow \neg P(y,y))$

The clauses are	$\{\neg P(y,a), \neg P(y,y)\}$	$\{P(y,y), P(y,a)\}$
Factoring yields	$\{\neg P(a,a)\}$	$\{P(a,a)\}$

Resolution yields the empty clause!

Slide 1003

A Non-Trivial Example

$$\exists x\,[P \to Q(x)] \land \exists x\,[Q(x) \to P] \to \exists x\,[P \leftrightarrow Q(x)]$$

Clauses are $\{P, \neg Q(b)\}$ $\{P, Q(x)\}$ $\{\neg P, \neg Q(x)\}$ $\{\neg P, Q(a)\}$

Resolve $\{P, \underline{\neg Q(b)}\}$ with $\{P, \underline{Q(x)}\}$ getting $\{P\}$

Resolve $\{\neg P, \underline{\neg Q(x)}\}$ with $\{\neg P, \underline{Q(a)}\}$ getting $\{\neg P\}$

Resolve $\{P\}$ with $\{\neg P\}$ getting $\square$

Implicit factoring: $\{P, P\} \mapsto \{P\}$ *Many other proofs!*

Slide 1004

Prolog Clauses and Their Execution

At most one positive literal per clause!

Definite clause $\{\neg A_1, \ldots, \neg A_m, B\}$ or $B \leftarrow A_1, \ldots, A_m.$

Goal clause $\{\neg A_1, \ldots, \neg A_m\}$ or $\leftarrow A_1, \ldots, A_m.$

Linear resolution: a program clause with last goal clause

Left-to-right through program clauses

Left-to-right through goal clause's literals

Depth-first search: backtracks, but still incomplete

Unification without occurs check: fast, but unsound!

Slide 1005

A (Pure) Prolog Program

```
parent(elizabeth,charles).
parent(elizabeth,andrew).

parent(charles,william).
parent(charles,henry).

parent(andrew,beatrice).
parent(andrew,eugenia).

grand(X,Z) :- parent(X,Y), parent(Y,Z).
cousin(X,Y) :- grand(Z,X), grand(Z,Y).
```

Slide 1006

Prolog Execution

```
                                           :- cousin(X,Y).
                               :- grand(Z1,X), grand(Z1,Y).
                :- parent(Z1,Y2), parent(Y2,X), grand(Z1,Y).
*                  :- parent(charles,X), grand(elizabeth,Y).
X=william                              :- grand(elizabeth,Y).
                  :- parent(elizabeth,Y5), parent(Y5,Y).
*                                        :- parent(andrew,Y).
Y=beatrice                                             :- □.
```

* = backtracking choice point

16 solutions including `cousin(william,william)`

and `cousin(william,henry)`

Slide 1007

The Method of Model Elimination

A Prolog-like method; complete for First-Order Logic

Contrapositives: treat clause $\{A_1, \ldots, A_m\}$ as m clauses

$$A_1 \leftarrow \neg A_2, \ldots, \neg A_m$$
$$A_2 \leftarrow \neg A_3, \ldots, \neg A_m, \neg A_1$$
$$\vdots$$

Extension rule: when proving goal P, may assume $\neg P$

A brute force method: efficient but no refinements such as subsumption

Slide 1008

A Survey of Automatic Theorem Provers

Hyper-resolution: Otter, Gandalf, SPASS, Vampire, . . .

Model Elimination: Prolog Technology Theorem Prover, SETHEO

Parallel ME: PARTHENON, PARTHEO

Higher-Order Logic: TPS, LEO

Tableau (sequent) based: LeanTAP, 3TAP, . . .

Slide 1009

Approaches to Equality Reasoning

Equality is *reflexive*, *symmetric*, *transitive*

Equality is *substitutive* over functions, predicates

- ***Use specialized prover: Knuth-Bendix,*** . . .
- ***Assert axioms directly***
- ***Paramodulation rule***

$$\frac{\{B[t], A_1, \ldots, A_m\} \qquad \{t = u, C_1, \ldots, C_n\}}{\{B[u], A_1, \ldots, A_m, C_1, \ldots, C_n\}}$$

Slide 1101

Modal Operators

W: set of *possible worlds* (machine states, future times, ...)

R: *accessibility relation* between worlds

(W, R) is called a *modal frame*

$\left.\begin{array}{l}\Box A \text{ means } A \text{ is \textit{necessarily true}} \\ \Diamond A \text{ means } A \text{ is \textit{possibly true}}\end{array}\right\}$ — in all **accessible** worlds

$\neg\Diamond A \simeq \Box\neg A$ A *cannot be true* $\iff$ A *must be false*

Slide 1102

Semantics of Propositional Modal Logic

For a particular frame (W, R)

An *interpretation* I maps the propositional letters to subsets of W

$w \Vdash A$ means A is true in world w $(w \in W)$

$$\begin{array}{lcl}
w \Vdash P & \iff & w \in I(P) \\
w \Vdash A \land B & \iff & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash \Box A & \iff & v \Vdash A \text{ for all } v \text{ such that } R(w, v) \\
w \Vdash \Diamond A & \iff & v \Vdash A \text{ for some } v \text{ such that } R(w, v)
\end{array}$$

Slide 1103

Truth and Validity in Modal Logic

For a particular frame (W, R), and interpretation I

$w \Vdash A$	means A is true in world w
$\models_{W,R,I} A$	means $w \Vdash A$ for all w in W
$\models_{W,R} A$	means $w \Vdash A$ for all w and all I

$\models A$ means $\models_{W,R} A$ for all frames; A is *universally valid*

. . . but typically we constrain R to be, say, **transitive**

All tautologies are universally valid

Slide 1104

A Hilbert-Style Proof System for K

Extend your favourite propositional proof system with

$$\text{Dist} \quad \Box(A \to B) \to (\Box A \to \Box B)$$

Inference Rule: *Necessitation*

$$\frac{A}{\Box A}$$

Treat $\Diamond$ as a definition

$$\Diamond A \stackrel{\text{def}}{=} \neg\Box\neg A$$

Slide 1105

Variant Modal Logics

Start with pure modal logic, K

Add axioms to constrain the accessibility relation:

T	$\Box A \to A$	(reflexive)	logic T
4	$\Box A \to \Box\Box A$	(transitive)	logic $S4$
B	$A \to \Box\Diamond A$	(symmetric)	logic $S5$

And countless others!

We shall mainly look at $S4$

Slide 1106

Extra Sequent Calculus Rules for $S4$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ (\Box l) \qquad \frac{\Gamma^* \Rightarrow \Delta^*, A}{\Gamma \Rightarrow \Delta, \Box A}\ (\Box r)$$

$$\frac{A, \Gamma^* \Rightarrow \Delta^*}{\Diamond A, \Gamma \Rightarrow \Delta}\ (\Diamond l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A}\ (\Diamond r)$$

$\Gamma^* \stackrel{\text{def}}{=} \{\Box B \mid \Box B \in \Gamma\}$ Erase non-$\Box$ assumptions

$\Delta^* \stackrel{\text{def}}{=} \{\Diamond B \mid \Diamond B \in \Delta\}$ Erase non-$\Diamond$ goals

Slide 1107

A Proof of the Distribution Axiom

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow B, A} \qquad \overline{B, A \Rightarrow B}}{A \to B, A \Rightarrow B}\ (\to l)}{A \to B, \Box A \Rightarrow B}\ (\Box l)}{\Box(A \to B), \Box A \Rightarrow B}\ (\Box l)}{\Box(A \to B), \Box A \Rightarrow \Box B}\ (\Box r)}{}$$

And thus $\Box(A \to B) \to (\Box A \to \Box B)$

Must apply $(\Box r)$ first!

Slide 1108

Part of an Operator String Equivalence

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Diamond A \Rightarrow \Diamond A}}{\Box\Diamond A \Rightarrow \Diamond A}\ (\Box l)}{\Diamond\Box\Diamond A \Rightarrow \Diamond A}\ (\Diamond l)}{\Box\Diamond\Box\Diamond A \Rightarrow \Diamond A}\ (\Box l)}{\Box\Diamond\Box\Diamond A \Rightarrow \Box\Diamond A}\ (\Box r)$$

In fact, $\Box\Diamond\Box\Diamond A \simeq \Box\Diamond A$ also $\Box\Box A \simeq \Box A$

The $S4$ operator strings are $\Box \quad \Diamond \quad \Box\Diamond \quad \Diamond\Box \quad \Box\Diamond\Box \quad \Diamond\Box\Diamond$

Slide 1109

Two Failed Proofs

$$\cfrac{\cfrac{\Rightarrow A}{\Rightarrow \Diamond A}\;(\Diamond r)}{A \Rightarrow \Box\Diamond A}\;(\Box r)$$

$$\cfrac{\cfrac{B \Rightarrow A \land B}{B \Rightarrow \Diamond(A \land B)}\;(\Diamond r)}{\Diamond A, \Diamond B \Rightarrow \Diamond(A \land B)}\;(\Diamond l)$$

Can extract a countermodel from the proof attempt

Slide 1201

Simplifying the Sequent Calculus

7 connectives *(or 9 for modal logic)*:

$$\neg \quad \land \quad \lor \quad \to \quad \leftrightarrow \quad \forall \quad \exists \qquad (\Box \quad \Diamond)$$

Left and right: so 14 rules *(or 18)* plus basic sequent, cut

Idea! Work in **Negation Normal Form**

Fewer connectives: $\land \quad \lor \quad \forall \quad \exists \qquad (\Box \quad \Diamond)$

Sequents need *one side only!*

Slide 1202

Simplified Calculus: Left-Only

$$\frac{}{\neg A, A, \Gamma \Rightarrow}\;(\mathit{basic}) \qquad \frac{\neg A, \Gamma \Rightarrow \qquad A, \Gamma \Rightarrow}{\Gamma \Rightarrow}\;(\mathit{cut})$$

$$\frac{A, B, \Gamma \Rightarrow}{A \land B, \Gamma \Rightarrow}\;(\land l) \qquad \frac{A, \Gamma \Rightarrow \qquad B, \Gamma \Rightarrow}{A \lor B, \Gamma \Rightarrow}\;(\lor l)$$

$$\frac{A[t/x], \Gamma \Rightarrow}{\forall x\, A, \Gamma \Rightarrow}\;(\forall l) \qquad \frac{A, \Gamma \Rightarrow}{\exists x\, A, \Gamma \Rightarrow}\;(\exists l)$$

Rule $(\exists l)$ holds *provided* x is not free in the conclusion!

Slide 1203

Left-Only Sequent Rules for $S4$

$$\frac{A, \Gamma \Rightarrow}{\Box A, \Gamma \Rightarrow}\ (\Box l) \qquad \frac{A, \Gamma^* \Rightarrow}{\Diamond A, \Gamma \Rightarrow}\ (\Diamond l)$$

$\Gamma^* \stackrel{\text{def}}{=} \{\Box B \mid \Box B \in \Gamma\}$ Erase non-$\Box$ assumptions

From 14 *(or 18)* rules to 4 *(or 6)*

Left-only system uses **proof by contradiction**

Right-only system is precisely dual

Slide 1204

Proving $\forall x\,(A \to B) \Rightarrow A \to \forall x\,B$

Left-only, NNF version: $A \land \exists x\,\neg B,\ \forall x\,(\neg A \lor B) \Rightarrow$

(x not free in A)

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{A,\ \neg B,\ \neg A \Rightarrow} \qquad \overline{A,\ \neg B,\ B \Rightarrow}}{A,\ \neg B,\ \neg A \lor B \Rightarrow}\ (\lor l)}{A,\ \neg B,\ \forall x\,(\neg A \lor B) \Rightarrow}\ (\forall l)}{A,\ \exists x\,\neg B,\ \forall x\,(\neg A \lor B) \Rightarrow}\ (\exists l)}{A \land \exists x\,\neg B,\ \forall x\,(\neg A \lor B) \Rightarrow}\ (\land l)$$

Slide 1205

Adding Unification

Rule $(\forall l)$ now inserts a **new** free variable:

$$\frac{A[z/x], \Gamma \Rightarrow}{\forall x\, A, \Gamma \Rightarrow}\ (\forall l)$$

Let unification instantiate any free variable

In $\neg A, B, \Gamma \Rightarrow$ try unifying A with B to make basic sequent

Updating a variable affects *entire proof tree*

What about rule $(\exists l)$? *Skolemize*!

Slide 1206

Skolemization from NNF

Follow tree structure; don't pull out quantifiers!

$[\forall y\, \exists z\, Q(y, z)] \land \exists x\, P(x)$ to $[\forall y\, Q(y, f(y))] \land P(a)$

Better to push quantifiers in (*miniscope*)

Proving $\exists x\, \forall y\, [P(x) \to P(y)]$

Negate; convert to NNF: $\forall x\, \exists y\, [P(x) \land \neg P(y)]$

Push in the $\exists y$: $\forall x\, [P(x) \land \exists y\, \neg P(y)]$

Push in the $\forall x$: $\forall x\, P(x) \land \exists y\, \neg P(y)$

Skolemize: $\forall x\, P(x) \land \neg P(a)]$

Slide 1207

A Proof of $\exists x\, \forall y\, [P(x) \to P(y)]$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y \mapsto f(z)}{P(y),\ \neg P(f(y)),\ P(z),\ \neg P(f(z)) \Rightarrow}\ \text{basic}}{P(y),\ \neg P(f(y)),\ P(z) \land \neg P(f(z)) \Rightarrow}\ (\land l)}{P(y),\ \neg P(f(y)),\ \forall x\, [P(x) \land \neg P(f(x))] \Rightarrow}\ (\forall l)}{P(y) \land \neg P(f(y)),\ \forall x\, [P(x) \land \neg P(f(x))] \Rightarrow}\ (\land l)}{\forall x\, [P(x) \land \neg P(f(x))] \Rightarrow}\ (\forall l)
$$

Unification chooses the term for $(\forall l)$

Slide 1208

A Failed Proof

Try to prove $\forall x\, [P(x) \lor Q(x)] \Rightarrow \forall x\, P(x) \lor \forall x\, Q(x)$

NNF: $\exists x\, \neg P(x) \land \exists x\, \neg Q(x),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow$

Skolemize: $\neg P(a) \land \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow$

$$
\cfrac{\cfrac{\cfrac{\cfrac{y \mapsto a}{\neg P(a),\ \neg Q(b),\ P(y) \Rightarrow} \qquad \cfrac{y \mapsto b???}{\neg P(a),\ \neg Q(b),\ Q(y) \Rightarrow}}{\neg P(a),\ \neg Q(b),\ P(y) \lor Q(y) \Rightarrow}\ (\lor l)}{\neg P(a),\ \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow}\ (\forall l)}{\neg P(a) \land \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow}\ (\land l)
$$

Slide 1209

The World's Smallest Theorem Prover?

```
prove((A,B),UnExp,Lits,FreeV,VarLim) :- !,
        prove(A,[B|UnExp],Lits,FreeV,VarLim).
prove((A;B),UnExp,Lits,FreeV,VarLim) :- !,
        prove(A,UnExp,Lits,FreeV,VarLim),
        prove(B,UnExp,Lits,FreeV,VarLim).
prove(all(X,Fml),UnExp,Lits,FreeV,VarLim) :- !,
        \+ length(FreeV,VarLim),
        copy_term((X,Fml,FreeV),(X1,Fml1,FreeV)),
        append(UnExp,[all(X,Fml)],UnExp1),
        prove(Fml1,UnExp1,Lits,[X1|FreeV],VarLim).
prove(Lit,_,[L|Lits],_,_) :-
        (Lit = -Neg; -Lit = Neg) ->
        (unify(Neg,L); prove(Lit,[],Lits,_,_)).
prove(Lit,[Next|UnExp],Lits,FreeV,VarLim) :-
        prove(Next,UnExp,[Lit|Lits],FreeV,VarLim).
```