Example Questions
Computer Systems Modelling

September 29, 2000

Simulation

1. In what contexts is simulation an appropriate technique for performance evaluation? When it is inappropriate?

2. How would you generate random variates for the Poisson distribution? Write pseudo code for a simulation component which models the Poisson arrival process of customers at a bank, given that the mean arrival rate is 40 customers per hour. What are the important events for this component (the generator).

3. How would you generate random variates from the mixed exponential distribution

$$f(x) = 1/3\lambda_1 e^{-\lambda_1 x} + 2/3\lambda_2 e^{-\lambda_2 x}$$

   where $\lambda_1 = 5, \lambda_2 = 10$, which describes the joint arrival process of two separate exponentially distributed streams of customers to a single queue?

4. Suppose that a simulation is constructed to estimate the response time, in milliseconds, of an interactive computer system. A number of repetitions are performed with the following results: $4.1, 3.6, 3.1, 4.5, 3.8, 2.9, 3.4, 3.3, 2.8, 4.5, 4.9, 5.3, 1.9, 3.7, 3.2, 4.1, 5.1$.

   Calculate the mean and variance of these results and hence derive bounds for a 95% confidence interval. You may wish to make use of the fact that $\mathbb{P}(Z > 1.96) > 0.025 > \mathbb{P}(Z > 1.97)$ where Z is a random variable with the unit normal distribution.

5. An experiment is performed to estimate the performance of a $M/M/1$ system with FIFO queueing. The response times of each of 1000 successive customers are recorded and the mean ($\bar{x}$) and variance (σ^2) of these numbers are calculated.

   A naïve student believes that $100(1-\alpha)\%$ confidence bounds for the mean can be derived by the expression

$$\bar{x} \pm \frac{z_{\alpha/2}\sigma}{\sqrt{1000}}$$

   where $z_{\alpha/2}$ is defined from the unit normal distribution in the usual manner.

   What mistake has this student made? Would their technique be valid if they were estimating the service time of this queue?

Operational analysis

1. In what kinds of context are the tools of operational analysis appropriate for performance evaluation? When are they inappropriate?

2. A printer and print queue are observed for 48 minutes during which time 16 jobs were completed. The mean queue length was 4 jobs. The printer could print 8 pages per minute. The mean job size was 12 pages.

   Assuming that the number of arrivals was equal to the number of completions, calculate (a) the throughput of the system in jobs-per-minute, (b) the mean residence time of a job, (c) the mean queuing time of a job and (d) the utilization of the printer.

3. An queueing network described as *open* and *feed forward* is to be analyzed using operational laws. The mean arrival rate into the system is known to be λ.

   (a) Describe the significance of the two phrases in italics.

   (b) The mean service times, visit counts and connectivity are known for each device. Why is it still impossible, in general, to calculate an upper bound on the residence time of a job in the network?

   (c) Suppose, however, that there are never more than five jobs in the network at once. Outline how you could derive an upper bound on the residence time.

4. Suppose that busses run according to a regular timetable with a fixed inter-arrival time of 10 minutes. What is the mean waiting time of a customer arriving at the bus stop?

   Suppose instead that bus inter-arrival times are exponentially distributed, still with mean 10 minutes. What now is the mean waiting time of an arriving customer?

   Derive a general relationship between the mean waiting time and the first and second moments of the inter-arrival distribution. *Hint: if the inter-arrival time pdf is $f(x)$ then note that the probability of encountering an interval of length τ is proportional to both $f(\tau)$ and to τ.*

Queueing theory

1. In what situations does queuing theory provide appropriate techniques for performance evaluation? When does it not?

2. Show for the M/M/1 queue that the probability that there are n or more customers in the system is given by ρ^n.

   Use this result to find a service rate μ such that, for given λ, n, α where $0 < \alpha < 1$, the probability of n or more customers in the system is given by α

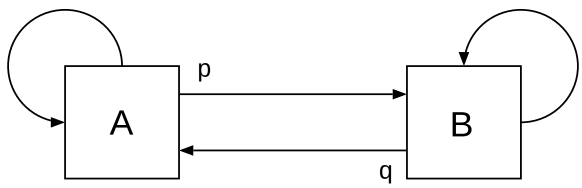
   Find a value of μ for an M/M/1 queue for which the arrival rate is 10 customers per second, and subject to the requirement that the probability of 3 or more customers in the system is 0.05

3. What is the average length of each idle period of an M/M/1 server, given its arrival rate λ service rate μ. What is the average length of a busy period?

4. Given an M/M/1/K queue with $\lambda = 10, \mu = 12, N = 15$ over what proportion of time are customers rejected from the queue? What is the effective arrival rate? What is the effective utilization of the server?

5. BrightnBrainy computer company's assembly machines break down according to a Poisson distribution with average rate 3 per hour. The company is considering the services of 2 service agents: I.M. Fast and I.M. Slow. The former can fix machines at a rate of 6 per hour, and charges £18 per hour whilst the latter can only service 4 per hour but charges only £10 per hour. If the cost of an idle machine is £30 per hour, which repair man should be used?

6. Repairing a computer takes 4 stages in sequence, namely removing the lid, finding the faulty part, replacing it, and reassembling the machine. Each step is independent and exponentially distributed with mean 3 minutes. What is the coefficient of variation of the repair time? Construct a Markov chain model of this system, assuming an infinite population of machines.



7. A closed queueing network (shown above) comprises two $M/M/1$ nodes, **A** and **B** between which n identical jobs circulate. The nodes have service rates μ_A and μ_B respectively. Upon completion at **A**, a job moves to **B** with probability p and otherwise it remains at **A**. Similarly, upon completion at **B**, a job moves to **A** with probability q.

   Derive a Markov-chain model of this system, explaining what each state in the model signifies and what transition rates exist between states.

Past exam questions

1. **1996 Paper 8 Q3.** A telephone exchange multiplexes 64Kb/s voice calls onto a 256Kb/s trunk line. New calls have an exponentially distributed inter-arrival process, with a mean of 20 seconds, and the call holding time is exponentially distributed with a mean of 60 seconds.

   (a) Draw a diagram of a Markov Chain which models the system, labelling the state transitions with their rates where appropriate. What is the necessary condition for stability of this system ? [5 marks]

   (b) Derive an expression for the probability that an arriving call finds k calls in progress, for $k \geq 0$, and thence calculate the probability that a caller finds the exchange engaged, given the parameters above. [15 marks]

2. **1996 Paper 9 Q3.** The Erlangian distribution E_r, with parameters (r, μ) is given by

$$f_X(x) = \frac{\lambda(\lambda x)^{r-1}}{(r-1)!}\mathrm{e}^{-\lambda x} \qquad x \geq 0$$

   (a) Given an $M/E_2/1$ queue, draw a Markov Chain describing the queueing system, and derive a formula for the variance of the service time distribution. Give an example of a queueing system in which the use of an Erlang service time distribution would be useful. [5 marks]

   (b) Describe how random variables from a given distribution function $f_X(x)$ can be sampled for use in a discrete event simulator. [3 marks]

   (c) Use your answer from the previous question to develop pseudo-code for a function in a discrete event simulator which when called returns a sampled value from the distribution function E_r. State any assumptions made and explain any arguments to the function. Comment on the efficiency of your code. [12 marks]

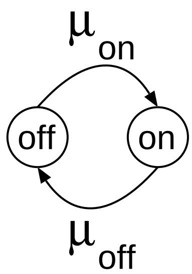
3. **1997 Paper 8 Q3.** A group of smart computer scientists decide to earn some money in the vacation by buying a punt and running a chaueured punt tour operation on the Cam. Based at Magdalene Bridge, they run tours upriver showing tourists the delights of the Backs. As those familiar with punt trips will know, journeys take an exponentially distributed time to complete, particularly when accompanied by strawberries and champagne. In this case our punt operators have measured that the average trip takes 30 minutes. Tourists (being tourists) behave rather randomly and arrive at the quay independently, according to Poisson distribution with mean 10 per hour. Each punt can accommodate six tourists, but departs as soon as there is at least one tourist wanting to take the tour.

   (a) Draw a Markov chain model for the punt tour operation, annotating it with the appropriate rates and state information. Briefly explain the diagram. [5 marks]

   (b) Propose a model which can safely be used as a worst-case approximation to the system above. Why is your model conservative? Using your new model, calculate the expected length of the queue of tourists. What constraints are required to ensure that the queuing system remains stable? [5 marks]

(c) Now assume that tourists do not join the queue if there are already six others waiting. If each tourist pays 5 per trip, and a punt costs 10 per hour to run, calculate the expected hourly prot of the punt company. Propose a scheme which would enable the entrepreneurs to increase their prots, but do not solve a model of the new system. What is the utilisation of the punts in the old and new schemes? [10 marks]

4. **1997 Paper 9 Q3.** The two-state Markovian on-o source is used in computer networking research as a model of bursty network traffic. The source is pictured in the figure below. When in the on state it generates fixed-sized network packets at a *constant* rate of 100 packets/second. Each set of packets generated in the on state is called a *burst*. When in the off state the source is silent. The residence time in each state is exponentially distributed. The *burstiness* of the source is dened as the ratio of its peak and mean rates.



A traffic modeller wants to write a simulation module which emulates this source, with the requirement that it transmits bursts of traffic with a mean length of 25 packets, and a burstiness of 20.

(*a*) Calculate the rates μ_{off} and μ_{on}. [10 marks]

(*b*) Briefly describe how to generate values which are distributed exponentially for use in the simulator. [3 marks]

(*c*) State Little's Law. Outline a proof of Little's Law with the aid of a diagram. [7 marks]

5. **1998 Paper 8 Q3.** Define the term *Markov Chain*. Why is the Markov property useful in modelling queueing systems? [5 marks]

Consider a birth-death queueing system with the following birth and death coefficients in which the state index represents the number of customers in the system:

$$\begin{array}{ll} \lambda_k = (k+2)\lambda & k = 0, 1, 2 \ldots \\ \mu_k = k\mu & k = 1, 2 \ldots \end{array}$$

All other coefficients are zero. Solve for p_k, the set of equilibrium probabilities for all states k, for $k = 0, 1, 2 \ldots$ State how you would find the average number of customers in the system. [15 marks]

6. **1998 Paper 9 Q3.** An $M/M/m$ queue has an arrival process with mean rate λ, and processes customers at a mean rate of μ.

(*a*) What are the distributions and parameters of the inter-arrival and service times of customers? [3 marks]

(*b*) Sketch an outline proof showing that the distribution of the departure process from the queue is the same as that of the arrivals process. [10 marks]

Briefly contrast analytical queueing analysis and discrete event simulation with regard to their fields of applicability and other important considerations for the systems modeller. [7 marks]

– *These questions are based substantially on an earlier exercise sheet by Simon Crosby*