Topics in concurrency 2020

Distributed games

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Lectures 10 - 16
Event structures

first, of the simplest kind, “prime event structures with binary conflict,” as originally introduced

From nets ...
Unfolding a (safe) Petri net:
An event structure
Event structures with binary conflict

An event structure with binary conflict comprises $E = (\mid E \mid, \leq, \#)$, consisting of
- a set $\mid E \mid$, of events
- partially ordered by $\leq$, the causal dependency relation, and
- a binary, irreflexive, symmetric relation $\#$ on $E$, the conflict relation, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$

if $e \geq e_0 \# e'_0 \leq e'$, then $e \# e'$

Say $e, e'$ are concurrent if $\neg(e \# e') \& e \not\leq e' \& e' \not\leq e$. 
States of an event structure

The configurations, \( C^\infty(E) \), of an event structure \( E \) consist of those subsets \( x \subseteq E \) which are

**Consistent:** \( \forall e, e' \in x. \neg(e \neq e') \) and

**Down-closed:** \( \forall e, e'. e' \leq e \in x \Rightarrow e' \in x. \)

For an event \( e \) the set \([e] = \text{def} \{ e' \in E \mid e' \leq e \}\) is a configuration describing the whole causal history of the event \( e \).

\( x \subseteq x' \), i.e. \( x \) is a sub-configuration of \( x' \), means that \( x \) is a sub-history of \( x' \).

\( (C^\infty(E), \subseteq) \) is a domain. Write \( C(E) \) for the set of finite configurations.
Example: Streams as event structures

-conflict (inconsistency) \[\rightarrow\text{ immediate causal dependency}\]
Simple parallel composition

000 \sim 001 \quad 010 \sim 011 \quad 110 \sim 111

00 \sim 01 \quad \vdots \sim 11

0 \sim 1

aaa \sim aab \quad aba \sim abb \quad bba \sim bbb

aa \sim ab \quad \vdots \sim bb

a \sim b
Another example
CCS operations on event structures?

\[ a.a.\texttt{nil} \parallel \overline{a}.\texttt{nil} \ ? \]
Product of event structures—an example

CCS parallel composition is derived from the product of event structures, \textit{e.g.}

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\times \\
\uparrow \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\uparrow \\
a \\
\uparrow \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\uparrow \\
c \\
\uparrow \\
\end{array}
\end{array} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
(b, *) \\
\uparrow \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\uparrow \\
(a, *) \\
\uparrow \\
\end{array}
\end{array} \\
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
(b, c) \\
\uparrow \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\uparrow \\
(a, c) \\
\uparrow \\
\end{array}
\end{array} \\
\end{array}
\begin{array}{c}
\begin{array}{c}
\uparrow \\
(*, c) \\
\uparrow \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}

The duplication of events with common images under the projections, as in the two events carrying \((b, *)\) can be troublesome!

\(\leadsto\) stable families, rigid families, ...
Convenient to use more general event structures based on consistency rather than binary conflict.

Often in examples event structures have a binary conflict and we can take advantage of this in diagrams.
Event structures

An event structure comprises \((E, \leq, \text{Con})\), consisting of

- a set \(E\), of events

- partially ordered by \(\leq\), the causal dependency relation, and

- a nonempty family \(\text{Con}\) of finite subsets of \(E\), the consistency relation,

which satisfy

\[
\{e' \mid e' \leq e\} \text{ is finite for all } e \in E, \\
\{e\} \in \text{Con} \text{ for all } e \in E, \\
Y \subseteq X \in \text{Con} \Rightarrow Y \in \text{Con}, \text{ and} \\
X \in \text{Con} \& e \leq e' \in X \Rightarrow X \cup \{e\} \in \text{Con}.
\]

Say \(e, e'\) are concurrent if \(\{e, e'\} \in \text{Con} \& e \not\leq e' \& e' \not\leq e\).
Examples

\[ \text{Con} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\} \} \]

The example on the right shows that the event structures defined by consistency are more general than those with binary conflict.
Configurations of an event structure

The configurations, \( C^\infty(E) \), of an event structure \( E \) consist of those subsets \( x \subseteq E \) which are

**Consistent:** \( \forall X \subseteq_{\text{fin}} x. \ X \in \text{Con} \) and

**Down-closed:** \( \forall e, e'. \ e' \leq e \in x \Rightarrow e' \in x. \)

For an event \( e \) the set \([e] =_{\text{def}} \{e' \in E \mid e' \leq e\}\) is a configuration describing the whole causal history of the event \( e \).

\( x \subseteq x' \), i.e. \( x \) is a sub-configuration of \( x' \), means that \( x \) is a sub-history of \( x' \).

If \( E \) is countable, \( (C^\infty(E), \subseteq) \) is a Berry dl domain (and all such so obtained). Finite configurations: \( C(E) \).
Maps of event structures

A map of event structures $f : E \to E'$ is a partial function $f : E \to E'$ such that for all $x \in C(E)$

$$fx \in C(E') \text{ and } e_1, e_2 \in x \& f(e_1) = f(e_2) \Rightarrow e_1 = e_2.$$ 

Note that when $f$ is total it restricts to a bijection $x \cong fx$, for any $x \in C(E)$. 

Maps preserve concurrency, and locally reflect causal dependency:

$$e_1, e_2 \in x \& f(e_1) \leq f(e_2) \text{ (both defined)} \Rightarrow e_1 \leq e_2.$$ 

A total map is rigid when it preserves causal dependency.
Computation paths

A **computation path**: a partial order \( p = (|p|, \leq_p) \) for which the set \( \{ e' \in |p| \mid e' \leq_p e \} \) is finite for all \( e \in |p| \).

Say a path \( p \) is **prime** if it has a top element \( \text{top}(p) \).

**Rigid inclusion** between paths \( p = (|p|, \leq_p) \) and \( q = (|q|, \leq_q) \):

\[
p \hookrightarrow q \text{ iff } |p| \subseteq |q| \land \forall e \in |p|, e' \in |q|. e' \leq_p e \iff e' \leq_q e.
\]
Rigid families

A **rigid family** is a non-empty set of *finite* paths $\mathcal{R}$ for which

$$p \rightarrow q \in \mathcal{R} \Rightarrow p \in \mathcal{R}.$$  

**Example.** $C(E)$ with configurations inheriting order from event structure $E$, ordered by inclusion.

But rigid families are more general ...
Event structures from rigid families

A rigid family $\mathcal{R}$ determines an event structure $\Pr(\mathcal{R})$ whose order of finite configurations is isomorphic to $(\mathcal{R}, \hookrightarrow)$.

The event structure $\Pr(\mathcal{R})$ has events the subset $P$ of prime paths of $\mathcal{R}$; causal dependency given by rigid inclusion; and consistency by compatibilty w.r.t. rigid inclusion.

There is an order isomorphism $\varphi_{\mathcal{R}} : (\mathcal{R}, \hookrightarrow) \cong (\mathcal{C}(\Pr(\mathcal{R})), \subseteq)$ given by

$$\varphi_{\mathcal{R}}(q) = \{ p \in P \mid p \hookrightarrow q \}$$

for $q \in \mathcal{R}$. Its inverse:

$$\theta_{\mathcal{R}}(x) = \bigcup x,$$

on $x \in \mathcal{C}(\Pr(\mathcal{R}))$. 

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The product of event structures $A$ and $B$

The set $|A| \times_* |B|$ is defined as
$$\{ (a, *) \mid a \in |A| \} \cup \{ (a, b) \mid a \in |A|, b \in |B| \} \cup \{ (*, b) \mid b \in |B| \}.$$  
It has partial projections $\pi_1, \pi_2$.

$$A \times B =_{\text{def}} \Pr(\mathcal{R})$$ where the rigid family $\mathcal{R}$ satisfies: $p \in \mathcal{R}$ iff

(i) $|p| \subseteq |A| \times_* |B|$;

(ii) $\pi_1|p| \in \mathcal{C}(A)$ and $\pi_2|p| \in \mathcal{C}(B)$ and the projections are locally injective on $|p|$ i.e., $\forall c, c' \in |p|$. $\pi_1(c) = \pi_1(c') \Rightarrow c = c'$ and $\forall c, c' \in |p|$. $\pi_2(c) = \pi_2(c') \Rightarrow c = c'$;

(iii) $\leq_p$ is the least transitive relation such that $c \leq_p c'$ if $\pi_1(c) \leq_A \pi_1(c')$ or $\pi_2(c) \leq_B \pi_2(c')$.  

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Augmentations

Let $E$ be an event structure with configuration $x$.

A path $p = (|p|, \leq_p)$ is an augmentation of $x$ iff $|p| = x$ and

$$\forall e \in |p|, e' \in |E| \cdot e' \leq_E e \Rightarrow e' \leq_p e.$$ 

Define a partial operation on augmentations

$\land : \text{Aug}(E) \times \text{Aug}(E) \rightarrow \text{Aug}(E)$

by taking

$$p \land q = \begin{cases} (|p|, (\leq_p \cup \leq q)^*) & \text{if } |p| = |q| \& (\leq_p \cup \leq q)^* \text{ is antisymmetric}, \\ \text{undefined} & \text{otherwise}. \end{cases}$$
Games and strategies as event structures
Event structures with polarity

An event structure with polarity comprises \((A, \text{pol})\) where \(A\) is an event structure with a polarity function \(\text{pol}_A : A \to \{+, -, 0\}\) ascribing a polarity \(+\) (Player), \(-\) (Opponent) or \(0\) (neutral) to its events.

A **game** shall be represented by an event structure with polarity in which no moves are neutral. (We can add winning conditions, payoff, ... later.)

**Notation.** For configurations \(x\) and \(y\), write \(x \subseteq^- y\) to mean inclusion in which all the intervening events are moves of **Opponent**. Write \(x \subseteq^+ y\) for inclusion in which the intervening events are neutral or moves of **Player**.
Operations on games

The dual, $A^\perp$, of a game $A$, comprises the same underlying event structure as $A$ but with a reversal of polarities.

A strategy in $A$ will be a strategy for Player; a strategy for Opponent — a counterstrategy — a strategy in $A^\perp$.

The simple parallel composition of games, and generally event structures with polarity, simply juxtaposes them.
Plays in a game $A$

A play (of Player) in $A$, an event structure with polarity, is an augmentation $p = (|p|, \leq_p)$ with $|p| \in C(A)$ s.t.

$$\forall a, a' \in |p|. \ a' \rightarrow_p a \ & \ pol_A(a') = + \ or \ pol_A(a) = - \ \Rightarrow \ a' \rightarrow_A a.$$  

Write $\text{Plays}(A)$ for the set of plays in $A$.

If $A$ is a game, the only augmentations allowed of a play $p$ additional to the immediate causal dependency of $A$ are those of the form $\ominus \rightarrow_p \ominus$. 
Let $A$ be an event structure with polarity.

A **bare strategy** in $A$ is a rigid family $\sigma \subseteq \text{Plays}(A)$ which is receptive, $p \in \sigma \& | p | \subseteq^* x \in \mathcal{C}(A) \Rightarrow \exists q \in \sigma. p \hookrightarrow q \in \sigma \& | q | = x$.

(Note $q$ is unique by courtesy.)

Write $\sigma : A$.

When $A$ is a game, say $\sigma$ is a **strategy**.
Strategies as maps of event structures

Proposition. If $\sigma : A$ then $\text{top} : \Pr(\sigma) \rightarrow A$ is a total map of event structures which preserves polarity and satisfies

- **courtesy**, $s' \rightarrow s$ and $\text{pol}(s') = +$ or $\text{pol}(s) = -$ in $\Pr(\sigma)$ implies $f_\sigma(s') \rightarrow_{A} f_\sigma(s)$ in $A$, and

- **receptivity**, $f_\sigma x \subseteq^\neg y$ in $C(A)$, for $x \in C(\Pr(\sigma))$, implies there is a unique $x' \in C(\Pr(\sigma))$ such that $f_\sigma x' = y$.

When $A$ is a game such total maps satisfying courtesy and receptivity are the concurrent strategies of Rideau and W, precisely those total maps preserving polarity for which the copycat strategy is identity w.r.t. composition of strategies. The strategies of this course correspond to the “rigid images” of R-W’s concurrent strategies; for which configurations map injectively to augmentations of $A$. 
From maps to strategies

Let \( f : S \to A \) be a total map of event structures which preserves polarity. Define \( \sigma(f) \) to be its rigid image, the rigid family

\[
\sigma(f) = \{ (fx, \leq fx) \mid x \in \mathcal{C}(S) \}
\]

where \( a' \leq_{fx} a \iff \exists s', s \in x. a' = f(s') \& a = f(s) \& s' \leq_{S} s \).

**Proposition.**
The rigid image is a strategy \( \sigma(f) : A \) if \( f \) is receptive and courteous.  
(The converse may fail: we may have \( \sigma(f) : A \) with \( f \) not receptive.)
Example of a strategy presented as a map

\[ S \quad \oplus \quad \sim \sim \quad \oplus \quad \text{configurations of } S = \text{“states of play”} \]

\[ \uparrow \quad \uparrow \]

\[ \ominus \quad \ominus \]

\[ \downarrow \]

\[ A \quad \oplus \quad \text{configurations of } A = \text{“positions of the game”} \]

\[ \ominus \quad \ominus \]

The strategy: answer either move of Opponent by the Player move.
Strategies between games

Let $A$ and $B$ be games.

A strategy **from** $A$ **to** $B$ is a strategy $\sigma : A \bot \parallel B$.

**Composition - the idea:**
Given $\sigma : A \bot \parallel B$ and $\tau : B \bot \parallel C$ we compose them by letting them play against each other in the shared game $B$. 
Example: copycat strategy from $A$ to $A$

$$\mathcal{CC}_A$$

$A \perp$ $A$

$\overline{a}_2$ $\ominus$ $\overline{a}_1$ $\ominus$ $a_2$ $\ominus$ $a_1$
Copycat in general

In copycat, \( \alpha_A : A^\perp \parallel A \), \( \alpha_A = \{(x, \leq_{\mathcal{C}A} | x) \mid x \in \mathcal{C}(\mathcal{C}A)\} \) where

\( \mathcal{C}A \) has the same events and polarity as \( A^\perp \parallel A \) but with causal dependency \( \leq_{\mathcal{C}A} \) given as the transitive closure of the relation

\[
\leq_{A^\perp \parallel A} \cup \{(\overline{c}, c) \mid c \in A^\perp \parallel A \text{ } \& \text{ } pol_{A^\perp \parallel A}(c) = +\}
\]

where \( \overline{c} \leftrightarrow c \) is the natural correspondence between \( A^\perp \) and \( A \). A finite subset is consistent iff its down-closure is consistent in \( A^\perp \parallel A \). Then,

\[
x \in \mathcal{C}(\mathcal{C}A) \text{ iff } x \in \mathcal{C}(A^\perp \parallel A) \text{ } \& \text{ } \forall c \in x. \text{ } pol_{A^\perp \parallel A}(c) = + \Rightarrow \overline{c} \in x.
\]
The Scott order

Defining a partial order — the Scott order — on configurations of $A$

\[ y \sqsubseteq_A x \iff y \supseteq^\pm \supseteq^- \cdots \supseteq^- \sqsubseteq^+ x \]

we obtain a factorization system \(((C(A), \sqsubseteq_A), \supseteq^-, \sqsubseteq^+), \text{i.e. } \exists! z. y \supseteq^- z\).

**Proposition**  \( z \in C(C'_A) \iff z_2 \sqsubseteq_A z_1. \)

The Scott order on configurations, and elements of rigid families, plays a surprising “undercover” role in the development of distributed/concurrent games.
Interaction of strategies — simple case

Let $A$ be a game. Let $\sigma : A$ be a strategy and $\tau : A^\perp$ a counterstrategy.

Their interaction is given by

$$\tau \circledast \sigma = \{ p \land q \mid p \in \sigma \land q \in \tau \land p \land q \text{ is defined} \} .$$

**Proposition.** The interaction is a bare strategy $\tau \circledast \sigma : A^0$ in which all moves are neutral.
Interaction in general

Let $A$, $B$, $C$ be games.

$$\bigotimes : \text{Plays}(B \perp \parallel C) \times \text{Plays}(A \perp \parallel B) \rightarrow \text{Plays}(A \perp \parallel B^0 \parallel C)$$

acts so

$$q \bigodot p = \text{def} \ (p \parallel y) \land (x_A \perp q)$$

where

$$| p | = x_A \perp x_B \quad \text{and} \quad | q | = y_B \perp y_C .$$

Let $\sigma : A \perp \parallel B$ and $\tau : B \perp \parallel C$. Define their interaction by

$$\tau \bigcirc \sigma = \{ q \bigodot p \mid p \in \sigma \& q \in \tau \& q \bigodot p \text{ is defined} \} : A \perp \parallel B^0 \parallel C .$$
Composition of strategies

Let $A$, $B$, $C$ be games. Define the projection

$$(\_\downarrow) : \text{Plays}(A^\perp \parallel B^0 \parallel C) \rightarrow \text{Plays}(A^\perp \parallel C) ,$$

of a play $p$ in $A^\perp \parallel B^0 \parallel C$, with $|p| = x_{A^\perp} \parallel x_B \parallel x_C$, to a play $p\downarrow$ in $A^\perp \parallel C$, to be the restriction of the order on $p$ to the set $x_{A^\perp} \parallel x_C$.

For plays $p \in \text{Plays}(A^\perp \parallel B)$ and $q \in \text{Plays}(B^\perp \parallel C)$, define

$q \odot p = \text{def} \ (q \otimes p) \downarrow .$

Let $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$. Define their composition by

$$\tau \odot \sigma = \{(q \odot p)\downarrow \mid p \in \sigma \& q \in \tau \& q \odot p \text{ is defined}\} .$$
A category of games and strategies

**Theorem.** Composition of strategies is associative and has identity the copycat strategy. *I.e.*, taking objects to be games and arrows from a game $A$ to a game $B$ to be strategies in the game $A \perp \parallel B$, with composition the composition of strategies, yields a category.

In fact, the category is *cpo-enriched*: inclusion between strategies is respected by composition and forms a cpo with bottom.

Often write $\sigma : A \rightarrow B$ when $\sigma : A \perp \parallel B$.

If $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ then $\tau \circ \sigma : A \rightarrow C$. 
Deterministic strategies

Let $A$ be an event structure with polarity. A bare strategy $\sigma : A$ is deterministic iff

$$p \rightarrow^+ q \& p \rightarrow r \text{ in } \sigma \Rightarrow \exists s \in \sigma. q \rightarrow s \& r \rightarrow s.$$  

The interaction of deterministic bare strategies is deterministic. The composition of deterministic strategies is deterministic.
Nondeterministic copycats

Take $A$ to consist of two events, one +ve and one −ve event, inconsistent with each other $\oplus \sim \ominus$. The construction $\mathcal{C}_A$:

$$
\begin{array}{c}
A^\perp \\
\ominus \\
\oplus \longrightarrow \oplus \\
A
\end{array}
$$

Recall $\mathcal{C}_A$ is built from $\mathcal{C}(\mathcal{C}_A)$. To see $\mathcal{C}_A$ is not deterministic, take $x$ to be the singleton set consisting \textit{e.g.} of the −ve event on the left and $s, s'$ to be the +ve and −ve events on the right.
Lemma. Let $A$ be an event structure with polarity. The copycat strategy $\alpha_A$ is deterministic iff $A$ satisfies

$$\forall x \in C(A), a, a' \in \mid A \mid. x \cup \{a\} \in C(A) \& x \cup \{a'\} \in C(A) \&$$

$$pol_A(a) = + \& pol_A(a') = -$$

$$\Rightarrow x \cup \{a, a'\} \in C(A). \quad \text{(Race-free)}$$

$\sim$ A subcategory of race-free games and deterministic strategies.
Example: a tree-like game

~~~ conflict (inconsistency)  →  immediate causal dependency
⊕ Player move               ⊖ Opponent move
Special cases

**Simple games** “Simple games” of game semantics arise when we restrict Games to objects and deterministic strategies which are ‘tree-like’—alternating polarities, with conflicting branches, beginning with opponent moves.

**Stable spans and stable functions** The sub-bicategory of Games where the events of games are purely +ve is equivalent to the bicategory of ‘stable spans’ used in nondeterministic dataflow; feedback is given by trace. When deterministic we obtain a sub-bicategory equivalent to Berry’s dl-domains and stable functions.

**Closure operators** A deterministic strategy in $A$ determines a closure operator on $C^\infty(A)^\top$ of Abramsky and Melliès.

**Receptive ingenuous strategies** Deterministic concurrent strategies coincide with the receptive ingenuous strategies of and Melliès and Mimram.
A language for strategies

**Types:** Games $A$, $B$, $C$, ... with operations $A^\perp$, $A\parallel B$, sums $\sum_{i\in I} A_i$, recursively-defined types, ...

A **term**

$$x_1 : A_1, \cdots, x_m : A_m \vdash t \dashv y_1 : B_1, \cdots, y_n : B_n,$$

denotes a strategy from $A_1\parallel \cdots \parallel A_m$ to $B_1\parallel \cdots \parallel B_n$.

**Idea:** $t$ denotes a strategy as a map $S \to \vec{A}^\perp\parallel \vec{B}$ and through this its rigid image, a strategy $\sigma : \vec{A}^\perp\parallel \vec{B}$. Supports duality of strategies: $\sigma : A^\perp\parallel B \cong (B^\perp)^\perp\parallel A^\perp$

The term $t$ describes witnesses, finite configurations of $S$, to a relation between finite configurations $\vec{x}$ of $\vec{A}$ and $\vec{y}$ of $\vec{B}$.
Copycat $x : A \vdash y \sqsubseteq_A x \dashv y : A$ and other terms “wiring” causal dependencies.

Composition

$$
\begin{array}{c}
\Gamma \vdash t \dashv \Delta \\
\Delta \vdash u \vdash H
\end{array}
\quad
\frac{}{\Gamma \vdash \exists \Delta. [t \parallel u] \dashv H}
$$

Sum $\sum_{i \in I} t_i$

Conjunction $t_1 \land t_2$

Feedback

\[
\begin{array}{c}
\Gamma \\
\hline
\Delta
\end{array}
\quad
\begin{array}{c}
t \\
\hline
A
\end{array}
\]

Recursion $\mu x : A. t$

Duplication $x : A \vdash \delta_A \vdash y : A, z : A$ provided ...

The above is linear; require games with symmetry and (co)monads for nonlinearity. Supports an operational semantics via bare strategies. Extends with probabilities ...
From strategies to probabilistic strategies

A strategy:

$$\sigma : A$$

**Aim**

(1) To endow $\sigma$, or equivalently $\Pr(\sigma)$, with probability, while

(2) taking account of the fact that in a strategy Player can’t be aware of the probabilities assigned by Opponent. (*E.g.* in ‘Matching pennies’)

*Causal independence between Player and Opponent moves will entail their probabilistic independence. Equivalently, probabilistic dependence of Player on Opponent moves will presuppose their causal dependence.*
Probabilistic event structures

A probabilistic event structure comprises an event structure $E = (E, \leq, \text{Con})$ together with a (normalized) continuous valuation, i.e. a function $w$ from the Scott open subsets of configurations $C^\infty(E)$ to $[0, 1]$ which is

(normalized) $w(C^\infty(E)) = 1$  
(strict) $w(\emptyset) = 0$

(monotone) $U \subseteq V \Rightarrow w(U) \leq w(V)$

(modular) $w(U \cup V) + w(U \cap V) = w(U) + w(V)$

(contiguous) $w(\bigcup_{i \in I} U_i) = \sup_{i \in I} w(U_i)$ for directed unions $\bigcup_{i \in I} U_i$.

Intuition: $w(U)$ is the probability of the result being in $U$.
A cts valuation extends to a probability measure on Borel sets of configurations.
A workable characterization: A probabilistic event structure comprises an event structure $E$ with a configuration-valuation $v : C(E) \to [0, 1]$ which satisfies

(normalized) $v(\emptyset) = 1$ and

(non-ve drop) $d_v^{(n)}[y; x_1, \ldots, x_n] \geq 0$, for all $n \in \omega$, and $y \subseteq x_1, \ldots, x_n$ in $C(E)$.

For $y \subseteq x_1, \ldots, x_n$ in $C(E)$,

$$d_v^{(n)}[y; x_1, \ldots, x_n] = \text{def } v(y) - \sum_{I} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i),$$

—the index $I$ ranges over $\emptyset \neq I \subseteq \{1, \ldots, n\}$ s.t. $\{x_i \mid i \in I\}$ is compatible.

(Sufficient to check the ‘drop condition’ for $y \subseteq x_1, \ldots, x_n$)

**Theorem.** Continuous valuations restrict to configuration-valuations.

A configuration-valuation extends to a unique continuous valuation on open sets, and that to a unique probabilistic measure on Borel subsets of configurations.

(The result holds in greater generality, for Scott domains)
Example

\[
\begin{array}{ccc}
\{a, b\} & \{a\} & \{b\}\\
1/4 & 1/2 & 1/4
\end{array}
\]

\[
\begin{array}{ccc}
\emptyset & \emptyset & \emptyset \\
1/4 & 1/4 & 1/4
\end{array}
\]
Probabilistic event structure with polarities

Let $E$ be an event structure in which (not necessarily all) events carry $+/\neg$. Write $x \sqsubseteq^p y$ if $x \subseteq y$ and no event in $y \setminus x$ has polarity $\neg$.

Now, a configuration-valuation is a function $v : C(E) \to [0, 1]$ for which

$$v(\emptyset) = 1, \quad x \subseteq^\neg y \Rightarrow v(x) = v(y), \text{ for all } x, y \in C(E),$$

and the “drop condition”

$$d^{(n)}_v[y; x_1, \cdots, x_n] \geq 0$$

for all $n \in \omega$ and $y \sqsubseteq^p x_1, \cdots, x_n$ in $C(E)$.

(Sufficient to check the ‘drop condition’ for $y \sqsubseteq^p x_1, \cdots, x_n$)

A probabilistic event structure with polarity comprises $E$ an event structure with polarity together with a configuration-valuation $v_E : C(E) \to [0, 1]$. 
Probabilistic strategies

Assume games are race-free, i.e. there is no immediate conflict between events of opposite polarity.

A probabilistic strategy in \( A \) comprises a strategy \( \sigma : A \) with a valuation \( \nu : \sigma \cong C(\Pr(\sigma)) \rightarrow [0, 1] \).

A race-free game \( A \) has a probabilistic copy-cat by taking \( \nu_{\alpha A} \) constantly 1 —this is a valuation as \( \alpha A \) is deterministic for race-free \( A \).

The interaction and composition valuations:

\[
v_{\tau \oplus \sigma}(r) = \sum \{v_{\tau}(q).v_{\sigma}(p) \mid q \oplus p = r\} \quad \text{for } r \in \tau \oplus \sigma \quad \text{and}
\]

\[
v_{\tau \odot \sigma}(r) = \sum \{v_{\tau}(q).v_{\sigma}(p) \mid p, q \text{ minimum s.t. } q \odot p = r\} \quad \text{for } r \in \tau \odot \sigma .
\]

\( \sim \) a category of probabilistic strategies on race-free games
A special case of composition without hiding: play-off

Given a probabilistic strategy \( v_\sigma, \sigma : A \) and counter-strategy \( v_\tau, \tau : A^\perp \) we obtain a probabilistic bare strategy \( \tau \otimes \sigma : A^0 \) with valuation

\[
v_{\tau \otimes \sigma} : \tau \otimes \sigma \rightarrow [0, 1].
\]

Via \( C(\Pr(\tau \otimes \sigma)) \cong \tau \otimes \sigma \) this makes \( \Pr(\tau \otimes \sigma) \) a probabilistic event structure so generates a measure \( \mu \) on \( C^\infty(A) \).

Adding \textbf{pay-off} as a random variable from \( C^\infty(A) \) get \textbf{expected pay-off} as the Lebesgue integral

\[
\int_{x \in C^\infty(A)} X(x) \, d\mu(x).
\]
Extensions

Winning conditions - determinacy

Coping with deadlock, a process language view

Imperfect information

Probabilistic and quantum strategies

Parallel causes - needed for a full account of probabilistic strategies

Continuous distributions - for probabilistic programming

Symmetry - for (co)monads and nonlinearity. Semantics of prog langs, applications

Determinacy and value theorems - programming optimal strategies

Learning strategies? ...