Petri nets

- Introduced in 1962 (though claimed to have been invented by 1939)
- Starting point: think of a transition system where a number of processes can be in a given state and then allow coordination
- **Conditions**: local components of state
- **Events**: transitions and coordination
- Allows study of concurrency of events, reasoning about causal dependency and how the action of one process might conflict with that of another
- The first of a range of models: event structures, Mazurkiewicz trace languages, asynchronous transition systems, . . .
- Many variants with different algorithmic properties and expressivity
$\infty$-multisets

Multisets generalise sets by allow elements to occur some number of times. $\infty$-multisets generalise further by allowing infinitely many occurrences.

$$\omega^\infty = \omega \cup \{\infty\}$$

Extend addition:

$$n + \infty = \infty \quad \text{for } n \in \omega^\infty$$

Extend subtraction

$$\infty - n = \infty \quad \text{for } n \in \omega$$

Extend order:

$$n \leq \infty \quad \text{for } n \in \omega^\infty$$

An $\infty$-multiset over a set $X$ is a function

$$f : X \to \omega^\infty$$

It is a multiset if $f : X \to \omega$. 
Operations on $\infty$-multisets

- $f \leq g$ iff $\forall x \in X. f(x) \leq g(x)$
- $f + g$ is the $\infty$-multiset such that
  \[ \forall x \in X. (f + g)(x) = f(x) + g(x) \]
- For $g$ a multiset such that $g \leq f$, 
  \[ \forall x \in X. (f - g)(x) = f(x) - g(x) \]
A general Petri net consists of
- a set of conditions $P$
- a set of events $T$
- a pre-condition map assigning to each event $t$ a multiset of conditions $\bullet t$
- a post-condition map assigning to each event $t$ an $\infty$-multiset of conditions $t^\bullet$
- a capacity map $\text{Cap}$ an $\infty$-multiset of conditions, assigning a capacity in $\omega^\infty$ to each condition
Dynamics

A marking is an $\infty$-multiset $M$ such that

$$M \leq \text{Cap}$$

giving how many tokens are in each condition.

The token game:

For $M, M'$ markings, $t$ an event:

$$M \xrightarrow{t} M' \iff \bullet t \leq M \quad \& \quad M' = M - \bullet t + t\bullet$$

An event $t$ has concession (is enabled) at $M$ iff

$$\bullet t \leq M \quad \& \quad M - \bullet t + t\bullet \leq \text{Cap}$$
Further examples

1

Cap: 5

2

Cap: 5

2

Cap: 5

2
Basic Petri nets

Often don’t need multisets and can just consider sets.

A basic net consists of
- a set of conditions $B$
- a set of events $E$
- a pre-condition map assigning a subset of conditions $\bullet e$ to any event $e$
- a post-condition map assigning a subset of conditions $e^\bullet$ to any event $e$ such that

$$\bullet e \cup e^\bullet \neq \emptyset$$

The capacity of any condition is implicitly taken to be 1:

$$\forall b \in B : \text{Cap}(b) = 1$$

A marking $\mathcal{M}$ is now a subset of conditions.

$$\mathcal{M} \xrightarrow{e} \mathcal{M}' \text{ iff } \bullet e \subseteq \mathcal{M} \text{ and } (\mathcal{M} \setminus \bullet e) \cap e^\bullet = \emptyset \text{ and } \mathcal{M}' = (\mathcal{M} \setminus \bullet e) \cup e^\bullet$$
Contact occurs in marking $M$ if there exists an event $e$ such that

$$\bullet e \subseteq M \quad (M \setminus \bullet e) \cap e^* \neq \emptyset$$

A basic net is safe if there is no marking reachable from the initial marking in which contact occurs.
CCS operations on basic nets

A safe Petri net semantics for CCS can be constructed by ‘surgery’ on the nets:

- Nil process
- Prefixing
- $p + q$
- $p \parallel q$