Probability and Computation: Problem Sheet 5 Solutions

You are encouraged to submit your solutions at student reception or by emailing them to luca.zanetti@cl.cam.ac.uk by 2pm Friday 28th of February

Question 1.

(i) Prove that for every $n \geq 2$ there is an unweighted, undirected n-vertex graph with conductance 1.

(ii) (Open-Ended Bonus Question): Can you characterise all graphs with that property?

Solution: *Item (i)*: Let G be the star graph (that is, a tree of diameter 2 – or equivalently, the tree where $n-1$ of its n vertices have degree 1). As explained in the lecture, the conductance can be alternatively defined as

$$\phi(G) = \min_{\substack{\emptyset \neq S \subsetneq V: \\ \mathrm{vol}(S) \leq \mathrm{vol}(V)/2}} \phi(S), \qquad \text{where} \quad \phi(S) = \frac{|E(S, V \setminus S)|}{\mathrm{vol}(S)}.$$

(Since G is unweighted, $w(S, V \setminus S) = |E(S, V \setminus S)|$.) Note that for any subset $S \subseteq V$ which does not include the center vertex denoted by c, $|E(S, V \setminus S)| = |S|$, $\mathrm{vol}(S) = |S| \cdot 1 = |S|$ and $\phi(S) = 1$. Furthermore, if $S = \{c\}$, then $|E(S, V \setminus S)| = n - 1$, $\mathrm{vol}(S) = n - 1 = \mathrm{vol}(V)/2$ and $\phi(S) = 1$. Any set S that includes c cannot contain more vertices since then the volume of S would exceed $\mathrm{vol}(V)/2$. Hence $\Phi(G) = 1$.

Item (ii): First, note that for $n = 3$, the complete graph is another example that has conductance 1, and thus, any connected graph with $n = 3$ has conductance 1.

Suppose $n \geq 4$ and $\phi(G) = 1$. Let u, v, w, t be 4 unique vertices such that $(u, v) \in E(G)$.

Since $(u, v) \in E(G)$, then $d(u) + d(v) > \mathrm{vol}(V)/2$, because otherwise $\phi(\{u, v\}) < 1$. Additionally, $d(u) + d(v) \leq \mathrm{vol}(V) - (d(w) + d(t))$. It implies that

$$d(w) + d(t) < \mathrm{vol}(V)/2.$$

Therefore, $(w, t) \notin E(G)$, otherwise for set $\phi(\{w, t\}) < 1$.

Thus, we have established that for every edge (u, v), all other edges of the graph have either u or v as one of its endpoints. Additionally, it is not possible to have edges (w_1, u) and (w_2, v) for $w_1 \neq w_2$, as then the statement would not hold for the edge (w_1, u). Thus, either u or v have to be an endpoint for all edges of G. Thus, G is a star graph.

Question 2. *Recall the 8-vertex graph from Lecture 11, slide 11, which has $1 - \lambda_2(P) \approx 0.13$. To get some idea of how small (or large) this value is, prove the following bounds on the conductance of any unweighted, connected, 3-regular graph with 8 vertices:*

(i) Show that for any such graph G, $\phi(G) \leq \frac{1}{2}$

(ii) Show that for any such graph G, $\phi(G) \geq 1/12$

(iii) Which lower and upper bounds on $1 - \lambda_2$, where λ_2 is associated with the transition matrix of the lazy walk on G, can you deduce from (i) and (ii) using Cheeger's inequality?

Solution: *Item (i)*: We apply a method that is similar to the guessing algorithm for MAX-CUT. Let $V = \{1, 2, \ldots, 8\}$. Let S be a random subset of size 4, and for any $1 \leq i \leq 7$, let $X_i = 1$ iff $i \in S$ and

$X_i = 0$ otherwise. As G is 3-regular, $\mathrm{vol}(S) = 4 \cdot 3 = 12$; this holds for any set S. Consider now the number of cut edges. By linearity of expectation,

$$\begin{aligned}
\mathbf{E}[\,|E(S, V \setminus S)|\,] &= \sum_{\{u,v\} \in E(G)} \mathbf{P}[\,(u \in S \cap v \notin S) \cup (u \notin S \cap v \in S)\,] \\
&= \sum_{\{u,v\} \in E(G)} \frac{4}{7} \\
&= \frac{4}{7} \cdot |E(G)| \\
&= \frac{4}{7} \cdot 4 \cdot 3 = \frac{48}{7}.
\end{aligned}$$

Now using the probabilistic method there exists a set S with $|E(S, V \setminus S)| \leq \lfloor \frac{48}{7} \rfloor = 6$. Since $\mathrm{vol}(S) = 12$, this implies $\phi(G) \leq 1/2$, and thus the conductance of the lazy/weighted graph is at most $1/4$.

(As a side remark, for the complete graph all sets S have the same number of edges, and a slight modification of the above approach would establish that the complete graph has the largest conductance among all regular graphs.)

Furthermore, the same bound can be also derived by taking S as a connected set of size 4. Since S is connected, it must have at least 3 edges inside and thus $|E(S, V \setminus S)| \leq 4 \cdot 3 - 2 \cdot 3 = 6$. Thus we again conclude that $\phi(G) \leq 1/2$ (for the unweighted graph). However, for larger degree, the bound obtained by the probabilistic method is superior as it remains $1/2$, while the bound derived from a connected set would converge towards 1.

Item (ii): Since the graph is connected, for any set S, $|E(S, V \setminus S)| \geq 1$. Furthermore, $\mathrm{vol}(S) \leq |S| \cdot 3$. Hence,

$$\phi(G) \geq \min_{1 \leq |S| \leq 4} \frac{1}{|S| \cdot 3} = \frac{1}{12}.$$

(Actually it might be possible to argue that any 3-regular graphs with 8 vertices is 2-connected, and if this was true the lower bound could be improved by a factor of 2.)

Item (iii): Recall that Cheeger's inequality states:

$$\frac{1 - \lambda_2}{2} \leq \phi(G) \leq \sqrt{2(1 - \lambda_2)},$$

and rearranging yields

$$\frac{\Phi(G)^2}{2} \leq 1 - \lambda_2 \leq 2\Phi(G).$$

Using our estimates from *(i)* and *(ii)* yields

$$\frac{1}{1152} \leq 1 - \lambda_2 \leq \frac{1}{2}.$$

Another try is to apply the bound from Lecture 10 (last line in slide 12). For that we need an upper bound on the diameter of G. It is not hard to prove that the diameter δ of any d-regular graph satisfies $\delta \leq 3n/d$. Therefore,

$$1 - \lambda_2 \geq \frac{1}{2 \cdot d \cdot (3n/d) \cdot n} = \frac{1}{6n^2} = \frac{1}{294},$$

which is a bit better, although probably still far from the truth (i.e., the worst possible 3-regular graph). We may conclude by saying that proving both general and "useful" lower bounds on $1 - \lambda_2$ is not easy...

Question 3. *Find the conductance of the following graphs*

1. *The n-vertex path.*
2. *The 2-dimensional $n \times m$ grid.*

3. *The complete binary tree of height h.*

Solution: We will give only approximate solutions and we won't prove that these sets actually minimise the conductance.

1. The conductance of the path is minimised by the a partition that cuts the path halfway through. It has conductance $\Theta(1/n)$
2. Let w.l.o.g. $n \leq m$. Then, consider a set containing the first $\lfloor m/2 \rfloor$ columns. It has conductance $\Theta(1/m)$.
3. Let r be the root of the tree and u, v its two children. Consider the subtree rooted at u. It has conductance $\Theta(2^{-h})$.

Question 4. *Prove the following: A finite, irreducible, aperiodic Markov chain with transition matrix P is reversible if and only if its transition probabilities satisfy*

$$P(j_1, j_2)P(j_2, j_3) \cdots P(j_{n-1}, j_n)P(j_n, j_1) = P(j_1, j_n)P(j_n, j_{n-1}) \cdots P(j_3, j_2)P(j_2, j_1),$$

for any sequence of states $j_1, \dots j_n$.

Solution: This result is known as *Kolmogorov's criterion.*

Since our Markov chain X_t is finite, irreducible and aperiodic it has a unique stationary distribution π, what is more X_t converges to π as $t \to \infty$.

($\Longrightarrow$): Assume that P is reversible, thus for any states $i, j \in \Omega$ we have

$$P(i, j) = \frac{\pi(j)P(j, i)}{\pi(i)}$$

It follows that

$$\begin{aligned}
&P(j_1, j_2)P(j_2, j_3) \cdots P(j_{n-1}, j_n)P(j_n, j_1) \\
&\quad = \left(\frac{\pi(j_2)P(j_2, j_1)}{\pi(j_1)}\right)\left(\frac{\pi(j_3)P(j_3, j_2)}{\pi(j_2)}\right) \cdots \left(\frac{\pi(j_n)P(j_n, j_{n-1})}{\pi(j_{n-1})}\right)\left(\frac{\pi(j_1)P(j_1, j_n)}{\pi(j_n)}\right) \\
&\quad = P(j_1, j_n)P(j_n, j_{n-1}) \cdots P(j_3, j_2)P(j_2, j_1),
\end{aligned}$$

($\Longleftarrow$): Assume that

$$P(j_1, j_2)P(j_2, j_3) \cdots P(j_{n-1}, j_n)P(j_n, j_1) = P(j_1, j_n)P(j_n, j_{n-1}) \cdots P(j_3, j_2)P(j_2, j_1),$$

holds for any sequence of states $j_1, \dots j_n$.

Now fix any two states x and y and observe that for any sequence of states $j_1, \dots, j_{n-1}$ we have,

$$\begin{aligned}
&P(y, x) \cdot \mathbf{P}[\, X_n = y, X_{n-1} = j_{n-1}, X_{n-2} = j_{n-2}, \dots, X_0 = x \mid X_0 = x \,] \\
&\quad = P(y, x) \cdot P(x, j_1)P(j_1, j_2) \cdots P(j_{n-1}, y) \\
&\quad = P(x, y) \cdot P(y, j_{n-1})P(j_{n-1}, j_{n-2}) \cdots P(j_1, x) \\
&\quad = P(x, y) \cdot \mathbf{P}[\, X_n = x, X_{n-1} = j_1, X_{n-2} = j_2, \dots, X_0 = y \mid X_0 = y \,].
\end{aligned}$$

Thus if we sum both sides of the equality above over all possible sequences $j_1, \dots, j_{n-1}$ we obtain

$$P(y, x) \cdot P^{(n)}(x, y) = P(x, y) \cdot P^{(n)}(y, x).$$

Now since for any $u, v \in \Omega$ we have $\lim_{n \to \infty} P^{(n)}(u, v) = \pi(v)$ we can take limits in n to conclude that

$$\pi(x)P(x, y) = \pi(y)P(y, x),$$

as claimed.

Question 5. *Let G be a connected graph and P be the transition matrix of the simple random walk on G.*

1. *Show that if -1 is an eigenvalue of P then the walk is periodic.*

2. *Show that if G is bipartite and μ is an eigenvalue of P then $-\mu$ is also an eigenvalue of G, and that μ and $-\mu$ have the same multiplicity.*

Solution:

1. Let f be any nonzero function such that $Pf = -f$. Notice that, for any u,

$$f(u) = -\sum_{v:\, u\sim v} \frac{f(v)}{d(u)}$$

Choose $u = \arg\max_w |f(w)|$. From the equation above it is clear that, for any $v \sim u$, $f(u) = -f(w)$. Since the graph is connected, we can deduce that $|f(x)| = |f(y)| \neq 0$ for any pair of vertices x, y. Moreover, if there exists an edge between x and y, it must hold that $f(x) = -f(y)$. Since f is nonzero everywhere, this condition can be only satisfied if G is bipartite, which implies that P is periodic.

2. We need to show that if f is such that $Pf = \mu f$, then there exists g such that $Pg = -\mu g$. Moreover, to show that multiplicities are preserved, we need to show that if pairwise orthogonal (w.r.t. $\langle \cdot, \cdot \rangle_\pi$ inner-product, where π is the stationary distribution of P) functions $f^1, \dots, f^k$ are such that $Pf^i = \mu f^i (1 \le i \le k)$, then there exist pairwise orthogonal $g^1, \dots, g^k$ such that $Pg^i = \mu g^i (1 \le i \le k)$.

   Since the graph is bipartite, we can write P (potentially after permuting the vertices) as a block matrix

$$P = \begin{pmatrix} 0 & P_2 \\ P_1 & 0 \end{pmatrix},$$

   where each block corresponds to one side of the bipartition. Let now $f = (f_1, f_2)$ be such that $Pf = \mu f$ (notice that this implies $Pf = (P_2 f_2, P_1 f_1) = (\mu f_1, \mu f_2)$). Let $g = (f_1, -f_2)$. then,

$$Pg = (-P_2 f_2, P_1 f_1) = (-\mu f_1, \mu f_2) = -\mu g.$$

   Moreover, assume we have pairwise orthogonal $f^1, \dots, f^k$ such that $Pf^i = \mu f^i (1 \le i \le k)$. Then, for any $1 \le i \le k$, we can construct $g^i = (f^i_1, -f^i_2)$. We have shown that with this construction $Pg^i = -\mu g^i$. We just need to show that, for any $i \neq j$, $g^i \perp g^j$. This can be shown as follows:

$$\langle g^i, g^j \rangle_\pi = \sum_x g^i(x) g^j(x) \pi(x) = \sum_x f^i(x) f^j(x) \pi(x) = \langle f^i, f^j \rangle_\pi.$$

Question 6. *Recall the definition of the ℓ_1-mixing time τ and the ℓ_2-mixing time τ_2. Prove that* ~~$\tau_2(\epsilon) \ge \tau(2\epsilon)$~~ $\tau_2(2\epsilon) \ge \tau(\epsilon)$ *for any $\epsilon \in (0, 1/2]$. (There was a typo.)*

Solution: This follows simply by the convexity of $(\cdot)^2$, which means we can apply *Jensen's inequality* to show that, for any function f, $\|f\|_{2,\pi} \ge \|f\|_1$.

Hints

Hint (Question 2(i)). *Use a randomised algorithm similar to the one for MAX-CUT to find a subset with conductance at most* $1/2$.

Hint (Question 4). *Recall the Convergence Theorem for finite Markov chains.*