Probability and Computation: Problem Sheet 5

You are encouraged to submit your solutions at student reception or by emailing them to luca.zanetti@cl.cam.ac.uk by 2pm Friday 28th of February

Question 1.

(i) Prove that for every $n \geq 2$ there is an unweighted, undirected n-vertex graph with conductance 1.

(ii) (Open-Ended Bonus Question): Can you characterise all graphs with that property?

Question 2. *Recall the 8-vertex graph from Lecture 11, slide 11, which has $1 - \lambda_2(P) \approx 0.13$. To get some idea of how small (or large) this value is, prove the following bounds on the conductance of any unweighted, connected, 3-regular graph with 8 vertices:*

(i) Show that for any such graph G, $\phi(G) \leq \frac{1}{2}$

(ii) Show that for any such graph G, $\phi(G) \geq 1/12$

(iii) Which lower and upper bounds on $1 - \lambda_2$, where λ_2 is associated with the transition matrix of the lazy walk on G, can you deduce from (i) and (ii) using Cheeger's inequality?

Question 3. *Find the conductance of the following graphs*

1. *The n-vertex path.*
2. *The 2-dimensional $n \times m$ grid.*
3. *The complete binary tree of height h.*

Question 4. *Prove the following: A finite irreducible, aperiodic Markov chain with transition matrix P is reversible if and only if its transition probabilities satisfy*

$$p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_n} p_{j_n j_{n-1}} \cdots p_{j_3 j_2} p_{j_2 j_1},$$

for any sequence of states $j_1, \ldots j_n$.

Question 5. *Let G be a connected graph and P be the transition matrix of the simple random walk on G.*

1. *Show that if -1 is an eigenvalue of P then the walk is periodic.*
2. *Show that if G is bipartite and μ is an eigenvalue of P then $-\mu$ is also an eigenvalue of G, and that μ and $-\mu$ have the same multiplicity.*

Question 6. *Recall the definition of the ℓ_1-mixing time τ and the ℓ_2-mixing time τ_2. Prove that $\tau_2(\epsilon) \geq \tau(2\epsilon)$ for any $\epsilon \in (0, 1/2]$.*

Hints

Hint (Question 2(i))**.** *Use a randomised algorithm similar to the one for MAX-CUT to find a subset with conductance at most* $1/2$*.*

Hint (Question 4)**.** *Recall the Convergence Theorem for finite Markov chains.*