Probability and Computation: Problem Sheet 4

You are encouraged to submit your solutions at student reception or by emailing them to jas289 by 2pm Friday 21th of February

Question 1. *Let X_n be the sum of n independent rolls of a fair die. Show that, for any $k \geq 2$,*

$$\lim_{n\to\infty} \mathbf{P}[\, X_n \text{ is divisible by } k\,] = \frac{1}{k}.$$

Question 2. *When the U bus arrives outside the Computer Lab, the next bus arrives in $1, 2, \ldots, 20$ minutes with equal probability. You arrive at the bus stop without checking the schedule, at some fixed time n.*

(i) How could you model X_n, the number of minutes until the next bus when you arrive at time n, as a Markov chain?

(ii) Buses have been coming and going all day so we can assume the chain has mixed when you arrive. What is the probability of waiting i minutes for a bus in relation to the Chain?

(iii) How long, on average, do you wait until the next bus arrives?

(iv) What is the standard deviation of this time?

Question 3. *Prove the following Lemma from class: For any probability distributions μ and η on a countable state space Ω*

$$\|\mu - \eta\|_{tv} = \frac{1}{2}\sum_{\omega\in\Omega} |\mu(\omega) - \eta(\omega)|.$$

Question 4. *This question asks you to prove lower bounds on the mixing time of some lazy random walks on graphs.*

1. *Let $G = (V_1 \cup V_2, E)$ be a graph made of two disjoint complete graphs of n vertices, supported respectively on V_1 and V_2, connected by a single edge. This is called the Barbell graph. Consider a lazy random walk on G. Prove that $t_{mix}(G) = \Omega(n^2)$ (recall from Lecture 8 that $t_{mix} = \tau(1/4)$).*
2. *Suppose now we add $s < n$ edges to the Barbell graph, where each edge has one endpoint in V_1 and the other endpoint in V_2. What happens to $t_{mix}(G)$?*
3. *Consider now a version of the Barbell graph where $|V_1| = n, |V_2| = \lfloor \log(n) \rfloor$ and there exists only an edge between V_1 and V_2. What is the mixing time of this graph?*

Question 5. *Let $\langle \cdot, \cdot \rangle_\pi$ be the inner product defined in the lecture. Show that it satisfies the following properties:*

Symmetry *For any $f, g \in \ell_2(\pi)$, $\langle f, g\rangle_\pi = \langle g, f\rangle_\pi$.*

Linearity *For any $f, g, h \in \ell_2(\pi)$ and $\alpha, \beta \in \mathbb{R}$, $\langle \alpha f + \beta g, h\rangle_\pi = \alpha\langle f, h\rangle_\pi + \beta\langle g, h\rangle_\pi$.*

Positive definiteness *For any $\underline{0} \neq f \in \ell_2(\pi)$, $\langle f, f\rangle_\pi > 0$.*

Do all these properties hold if π is not always positive?

Question 6. *Given a matrix M such that $Mf = \lambda f$ (i.e., f is an eigenvector with eigenvalue λ of M), prove that $M^k f = \lambda^k f$*

Hints

Hint (Question 1). *At face value X_n is an (infinite) Markov chain on $\mathbb{N}$. We would like to consider it as a finite Markov chain, reduction $\mod m$ (for some suitable m) will help us achieve this.*

Hint (Question 4(i)). *First prove that, for a distribution p such that $\sum_{u \in V_2} p(u) \leq \epsilon$ for some small $\epsilon \geq 0$, $\|p - \pi\|_{TV} \geq \frac{1}{2} - \epsilon$, where π is the stationary distribution of a random walk on the Barbell graph. Use this fact to obtain a lower bound on the mixing time (think how many steps we need so that $\sum_{u \in V_2} p^t(u) > \epsilon$, where p^t is the random walk distribution after t steps).*