Probability and Computation: Problem Sheet 3

You are encouraged to submit your solutions at student reception or by emailing them to jas289 by 2pm Friday 14th of February

Question 1 (Schöning: tighter analysis). *Use the following version of Schöning's Algorithm:*

> *(1) Start with a random truth assignment.*
>
> *(2) Repeat up to 3n times, terminating if all clauses are satisfied:*
>
> *(a) Choose an arbitrary clause that is not satisfied*
>
> *(b) Choose one of it's literals UAR and switch the variables value.*
>
> *(3) If a valid solution is found* **`return`** *it. O/W* **`return`** *unsatisfiable*

(i) Fix some satisfying assignment α. Let A_k be the event that the random assignment from step (1) disagrees with α on exactly k literals/variables. What is $\mathbf{P}[\, A_k \,]$?

(ii) Let P_k be the probability that we make $\leq k$ incorrect steps within our first $3k$ steps. Prove

$$P_k \geq \binom{3k}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{2k}.$$

(iii) Recall Stirling's inequality $\sqrt{2\pi} \leq \frac{n!}{n^{n+1/2}e^{-n}} \leq e$, and show that $P_k \geq \frac{2^{-k}}{3\sqrt{k}}$.

(iv) Show that if a solution exists, Schöning's Algorithm succeeds with probability at least $\left(\frac{3}{4}\right)^n/(3\sqrt{n})$

(v) Deduce a bound on the time to find a solution w.h.p. using Schöning's Algorithm as above.

Question 2. *Consider the following Markov Chains*

$$A = \begin{pmatrix} 0 & 1/9 & 2/9 & 2/3 \\ 1/7 & 1/7 & 5/7 & 0 \\ 2/9 & 5/9 & 0 & 2/9 \\ 3/5 & 0 & 1/5 & 1/5 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 2/5 & 0 & 0 & 3/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 1/3 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \end{pmatrix}$$

(i) Which of the above are irreducible?

(ii) Which of the above are reversible?

(iii) Calculate the stationary distribution of the reversible irreducible chain(s) above.

(iv) In lectures we showed that any finite irreducible chain has a unique stationary distribution. Give an example of a (finite :p) reducible chain with more than one stationary distribution.

Question 3. *Recall that an undirected weighted graph $G = (V, E, w)$ is an undirected graph with a weight function $w : E \to \mathbb{R}_+$ which is positive and symmetric, that is for any $ij \in E$, $w(ij) = w(ji) > 0$.*

(i) Let P be a Markov chain. Show that P is reversible if and only if P is a simple random walk on an undirected weighted graph G.

(ii) Show that a simple random walk on an undirected graph $G = (V, E, w)$ has stationary distribution

$$\pi(x) = \frac{\sum_{xy\in E} w(xy)}{2\sum_{e\in E} w(e)}, \qquad \text{for all } x \in V.$$

(iii) Given a reversible Markov chain P, show that P is irreducible if and only if the associated undirected weighted graph G is connected.

Question 4. *Recall that a probability vector (distribution) is a non-negative real vector whose elements sum to 1. A stochastic matrix is a real square matrix, where each row is a probability vector. Observe every Stochastic matrix gives rise to a Markov chain and vice versa.*

(i) Let $\nu \in \mathbb{R}^n_+$ be a probability vector and $M \in \mathbb{R}^{n\times n}_+$ be a stochastic matrix. Show that νM is a probability vector.

A doubly stochastic matrix is a real square matrix, where each row and column is a probability vector.

(ii) Prove that the uniform distribution is stationary for any Markov chain whose transition matrix is doubly stochastic.

Question 5. *Show that if P is the transition matrix of a reversible Markov chain then the matrix P^t also defines a reversible Markov chain.*

Question 6. *Consider the Complete graph K_n which is the graph on n vertices where each pair of vertices is connected by an edge. Let $x, y \in V$ where $x \neq y$.*

(i) Show that $\mathbf{E}_x\left[\tau_y^+\right] = n - 1$.

(ii) What is the distribution of τ_y^+?

Question 7. *State j is accessible from state i if, for some integer $n \geq 0$, $P^n_{i,j} > 0$. If two states i and j are accessible from each other, we say that they communicate and we write $i \sim j$. Prove that communicating relation $\sim$ defines an equivalence relation.*

Question 8. *Prove rigorously the claim made in lecture that the expected time for RAND 2-SAT to find a given solution is at most the hitting time $h(0, n)$ of the random walk on a path.*