Probability and Computation: Problem sheet 2

You are encouraged to submit your solutions at student reception or by emailing them to nr454 by Friday 07th of February

Algorithms

Question 1. *In this question we consider the following* NP*-complete problem called* VERTEX-COVER

Instance: *A graph* $G = (V, E)$.   ***Output:*** *Subset* $S \subseteq V$ *such that each edge has an end in* S *and* $\|S\|$ *is minimized.*

(i) Can you come up with a simple randomised algorithm for VERTEX-COVER*?*

(ii) Is this algorithm correct? (That is, does it always output a valid vertex-cover S*)*

(iii) Can you show that, for any graph G*, the expected size of the vertex cover* S *produced by your algorithm is at most twice the size of an optimal one?*

Conditional Expectation

Question 2. *Show properties* $2 - 6$ *of slide* 7 *of Lecture* 5.

Question 3. *Let* $X_1, \ldots, X_n$ *be independent discrete random variables and let* $Z = f(X_1, \ldots, X_n)$ *for some function* f*. Prove that*

$$\mathbf{E}[\, Z | X_1, \ldots, X_i \,] = \sum_{x_{i+1}, \ldots, x_n} f(X_1, \ldots, X_i, x_{i+1}, \ldots, x_n) \mathbf{P}[\, X_i = x_{i+1}, \ldots, X_n = x_n \,]$$

Question 4. *Conditional Variance. Define the conditional variance of* Y *given* X *as*

$$\mathbf{Var}\,[\, Y|X \,] = \mathbf{E}\big[\, (Y - \mathbf{E}[\, Y|X \,])^2 | X \,\big] \, .$$

(i) Prove that $\mathbf{Var}\,[\, Y \,] = \mathbf{E}[\, \mathbf{Var}\,[\, Y|X \,] \,] + \mathbf{Var}\,[\, \mathbf{E}[\, Y|X \,] \,]$

(ii) Consider n *bins and a random number* M *of balls, where* $\mathbf{E}[\, M \,] = \mu$ *and* $\mathbf{Var}\,[\, M \,] = \sigma^2$*. Compute the variance of the number of balls that are assigned to the first bin.*

Question 5. *Consider a coin that shows head with probability* p*. What is the expected number of flips required to observe a run of* n *consecutive heads?*

Hint.

Q5: *Recall how we deduce the expectation of a geometric random variable in class.*