Probability and Computation: Mock Exam

Question 1. *Consider the balls into bins problem where m balls are assigned uniformly and independently at random to n bins, where $m > n$. Let X be the number of empty bins.*

(a) Compute $\mathbf{E}[\,X\,]$.

(b) Prove that X is Liptschitz as a function of the bin number to which each ball is assigned.

(c) Use McDiarmid's inequality to derive an upper bound for $\mathbf{P}[\,X > \mathbf{E}[\,X\,] + t\,]$ *provided* $t > 0$.

(d) Find a better bound for $\mathbf{P}[\,X > \mathbf{E}[\,X\,] + t\,]$ *by expressing X as a function of something different.*

Question 2.

(a) Fill the missing entries in the matrix below so that it represents the transition matrix of a reversible Markov chain:

$$\begin{pmatrix} 0 & 3/4 & 1/4 & 0 & \dots \\ \dots & 0 & \dots & 0 & 0 \\ \dots & 4/7 & 0 & 2/7 & 0 \\ 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

(b) Find the stationary distribution of your matrix. Is the corresponding Markov chain irreducible? Is it aperiodic? Explain your answers.

(c) Let P be a transition matrix of a Markov chain on state space Ω. Let π be a probability distribution satisfying the following equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x) \quad \forall x, y \in \Omega.$$

Prove that π is a stationary distribution for P.

Question 3. *A matching in a graph is a set of edges without common vertices. In the Maximum Bipartite Matching problem, we are given a bipartite graph $G(L \cup R, E)$ (without multiple edges), and we want to find a matching of maximum cardinality. Consider the following randomised algorithm for this problem: Each edge is selected independently with probability p. All edges that have common endpoints are discarded. Assume that the bipartite graph has $|L| = |R| = n$ and that every vertex has degree 3.*

(a) What is the expected cardinality of the matching returned by the algorithm as a function of p?

(b) Find the value of p that maximises the expected cardinality of the matching. What is the expected cardinality of the matching in this case?

(c) Assume now the graph is regular of degree $d \geq 3$, not necessarily constant. Would you choose a constant value of p or a value that depends on d and/or n? Explain your choice.