Lecture 3: Concentration Inequalities

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A question from last class

Last lecture a student asked me how it is possible to have random variables X and Y such that $X \leq Y$.

This is one of the reasons to recall that random variables are actually functions.
Hence, when we say that $X \leq Y$ what we actually mean is that the event

$$\mathcal{E} = \{\omega \in \Omega : X(\omega) \leq Y(\omega)\}$$

is such that $\mathbf{P}[\,\mathcal{E}\,] = 1$.

It is very important for you to understand this notion as it is used in the proof of Markov inequality, but it will appear in some examples later

A question from last class

For example: Suppose you flip a fair coin 100 times. Here the outcome space is the values of the 100 coins, i.e $\Omega = \{H, T\}^{100}$, and the probability of each particular outcome is $(1/2)^{100}$. Define the random variables

$$X(\omega) = \begin{cases} \min\{i : \omega_i = H\} & \omega \neq (T, \ldots, T) \\ 0 & \omega = (T, \ldots, T) \end{cases},$$

and

$$Y(\omega) = \begin{cases} \max\{i : \omega_i = H\} & \omega \neq (T, \ldots, T) \\ 0 & \omega = (T, \ldots, T) \end{cases}.$$

Hence $X \leq Y$.

In practice, we don't describe random variables like that as it is too messy.

We would just say X is the position of the first Head and Y is the position of the last Head (and if there are no heads, we just define X and Y as 0).

Today class: Chernoff bounds

Recall the Markov and Chebyshev inequalities from the previous lecture

Markov Inequality

If X is a non-negative random variable and $a > 0$, then

$$\mathbf{P}[X \geq a] \leq \mathbf{E}[X]/a.$$

Chevyshev Inequality

If X is a random variable and $a > 0$, then

$$\mathbf{P}[|X - \mathbf{E}[X]| \geq a] \leq \mathbf{Var}[X]/a^2.$$

Let $f : \mathbb{R} \to [0, \infty)$ and increasing, then $Y = f(X) \geq 0$, and thus

$$\mathbf{P}[X \geq a] \leq \mathbf{P}[f(X) \geq f(a)] \leq \mathbf{E}[f(X)]/f(a).$$

Similarly, if $g : \mathbb{R} \to [0, \infty)$ and decreasing, then $Y = g(X) \geq 0$, and thus

$$\mathbf{P}[X \leq a] \leq \mathbf{P}[g(X) \geq g(a)] \leq \mathbf{E}[g(X)]/g(a).$$

By choosing an appropriate function we can obtain inequalities that are much sharper than the Markov and Chevyshev Inequality.

Example: coin flip

Consider n fair coins and let X be the total number of head. In an experiment we expect to see around $n/2$ heads. Can we justify that? Let $\delta > 0$

- Markov inequality :

$$\mathbf{P}[\,X \geq (1+\delta)(n/2)\,] \leq \frac{\mathbf{E}[\,X\,]}{(1+\delta)(n/2)} = \frac{n/2}{(1+\delta)(n/2)} = \frac{1}{1+\delta}$$

Not good! Independent of n

- Chebychev inequality :

$$\begin{aligned}\mathbf{P}[\,(X - n/2) \geq \delta(n/2)\,] &\leq \mathbf{P}\left[\,(X - n/2)^2 \geq (\delta(n/2))^2\,\right] \\ &\leq \frac{4\mathbf{Var}\,[\,X\,]}{\delta^2 n^2} = \frac{1}{\delta^2 n}\end{aligned}$$

Better! Linear in n

Markov and Chebychev use the first and second moment of the random variable. Can we keep going?

- Yes.

We can consider first, second, third and more moments! that is the basic idea behind the Chernoff Bounds

Outline

Chernoff Bounds

Balls into Bins

Proof of Chernoff Bounds

Randomised QuickSort

Chernoff Bounds

Chernoff Bounds

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[\,X\,] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[\,X \geq (1+\delta)\mu\,] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}.$$

and for $t > \mu$ it holds that

$$\mathbf{P}[\,X \geq t\,] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t},$$

Example: Coin Flip

Consider n fair coins and let X be the total number of head. Then

$$\mathbf{P}[\,X \geq (1+\delta)(n/2)\,] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{n/2}$$

Node that the above expression equals 1 only for $\delta = 0$, and then it gives a value strictly less than 1 (**Ex: check this!**). Note as well the inequality is exponential in n, (for fixed δ) i.e. much better than Chebychev inequality.

Example: Coin Flip

Consider 100 independent coin flips . We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.

- Markov's inequality : $X = \sum_{i=1}^{100} X_i$, $X_i \in \{0, 1\}$ and $\mathbf{E}[X] = 100 \cdot \frac{1}{2} = 50$.

$$\mathbf{P}[X \geq 3/2 \cdot \mathbf{E}[X]] \leq 2/3 = 0.666.$$

- Chebyshev's inequality : $\mathbf{Var}[X] = \sum_{i=1}^{100} \mathbf{Var}[X_i] = 100 \cdot (1/2)^2 = 25$.

$$\mathbf{P}[|X - \mu| \geq t] \leq \frac{\mathbf{Var}[X]}{t^2},$$

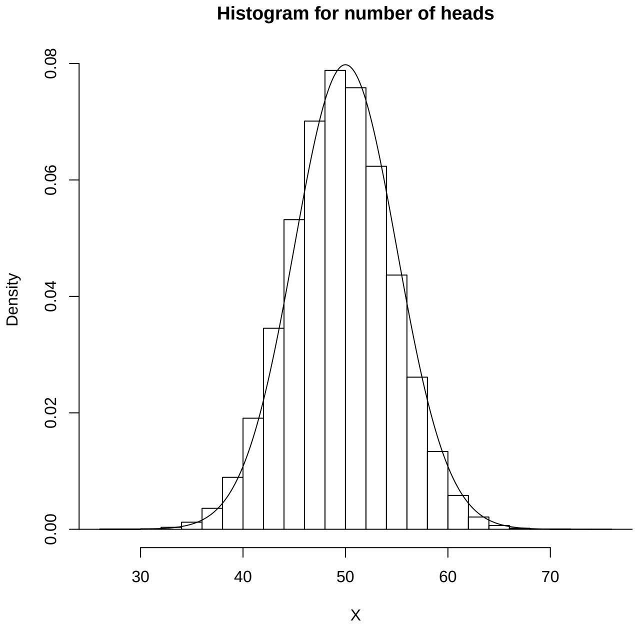
  and plugging in $t = 25$ gives an upper bound of $25/25^2 = 1/25 = 0.04$, much better than what we obtained by Markov's inequality.

- The Chernoff bound (first) with $\delta = 1/2$ gives:

$$\mathbf{P}[X \geq 3/2 \cdot \mathbf{E}[X]] \leq \left(\frac{e^{1/2}}{(3/2)^{3/2}} \right)^{50} = 0.004472$$

- the exact probability is $0.00000028\ldots$

Chernoff bound yields a more accurate result but needs independence!


Histogram for number of heads
Density
0.00
0.02
0.04
0.06
0.08
30
40
50
60
70
X

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Example: Balls into Bins

Balls into Bins is another important problem in the design and analysis of randomised algorithm.

Balls into Bins Problem

You have n boxes and m balls. Each ball is allocated in a box uniformly at random.

In the context of Computer Science there are several interpretations

1. Boxes are a Hash Table, balls are items
2. Boxes are processors and balls are jobs
3. Boxes are servers and ball are queries

Exercise: Think about the relation between the Balls into Bins problem and the Coupon Collector Problem.

Balls into Bins Problem

You have n boxes and m balls. Each ball is allocated in a box uniformly at random.

Balls into Bins question

How large is the maximum load?

- Focus on one bin. Let X_i the indicator variable that indicates if ball i is assigned to this bin. The total balls in the bin is given by $X = \sum_i X_i$. Note that $p_i = \mathbf{P}[\, X_i = 1\,] = 1/n$.
- Suppose that $m = 2n \log n$, then $\mu = \mathbf{E}[\, X\,] = 2 \log n$

$$\mathbf{P}[\, X \geq t\,] \leq e^{-\mu}(e\mu/t)^t$$

- By the Chernoff Bound,

$$\mathbf{P}[\, X \geq 6 \log n\,] \leq e^{-2\log n}\left(\frac{2e\log n}{6\log n}\right)^{6\log n} \leq e^{-2\log n} = n^{-2}$$

Example: Balls into Bins

- Let $\mathcal{E}_j$ be the event that bin j receives more than $6 \log n$ balls.
- We are interested in the probability that at least one bin receives more than $6 \log n$ balls
- This is the event $\cup_{j=1}^{n} \mathcal{E}_j$
- By the Union Bound, $\mathbf{P}\left[\cup_{j=1}^{n} \mathcal{E}_j\right] \leq \sum_{j=1}^{n} \mathbf{P}[\mathcal{E}_j] \leq n \times n^{-2} = n^{-1}$
- Therefore *whp*, no bin receives more than $6 \log n$ balls
- Note that the max loaded bin receives at least $2 \log n$ balls (**why?**). So our bound is pretty sharp.

whp stands for *with high probability* :
An event $\mathcal{E}$ (that implicitly depends on an input parameter n) occurs whp if $\mathbf{P}[\mathcal{E}^c] \to 0$ as $n \to \infty$.
This is a very standard notation in randomised algorithms but it may very from author to author. Be careful!

Example: Balls into Bins

Consider now the case $m = n$, i.e. same number of balls and bins.
Using the Chernoff Bounds

$$\mathbf{P}[\,X > t\,] \leq e^{-1}\left(\frac{e}{t}\right)^t \leq \left(\frac{e}{t}\right)^t$$

$$\boxed{\mathbf{P}[\,X \geq t\,] \leq e^{-\mu}(e\mu/t)^t}$$

By setting $t = 4\log n/\log\log n$, we obtain that $\mathbf{P}[\,X > t\,] \leq n^{-2}$. Indeed

$$\left(\frac{e\log\log n}{4\log n}\right)^{4\log n/\log\log n} = \exp\left(\frac{4\log n}{\log\log n}\cdot\log\left(\frac{e\log\log n}{4\log n}\right)\right) \tag{1}$$

The term inside the exponential is

$$\left(\frac{4\log n}{\log\log n}\cdot\left(\log(4/e) + \log\log\log n - \log\log n\right)\right) \leq \left(\frac{4\log n}{\log\log n}\left(-\frac{1}{2}\log\log n\right)\right)$$

obtaining that $\mathbf{P}[\,X > t\,] \leq n^{-4/2} = n^{-2}$.

This inequality only works for large enough n

Example: Balls into Bins

We just proved that

$$\mathbf{P}[\, X > 4 \log n / \log \log n \,] \leq n^{-2},$$

thus by the union bound , no bin receives more than $O(\log n / \log \log n)$ balls with probability at least $1 - 1/n$.

- You will see in your Exercise class how to prove that whp at least one bin receives at least $c \log n / \log \log n$ balls, for some c. You will learn a proof strategy called the Probabilistic Method.

Example: Balls into Bins

Conclusions

- If the number of balls is $2 \log n$ times n (the number of bins), then to distribute balls at random is a good algorithm
  - This is because the worst case load is whp. $6 \log n$, while the expected number of balls is $2 \log n$
- For the case $n = m$, the algorithm is not good, because the maximum load is whp. $\Theta(\log n / \log \log n)$, while the expected number of balls per bin is 1.
- For the case $n = m$, we can improve the balls into bin process by sampling two bins in each step, then assigning the ball into the bin with lesser load. This gives a maximum load $\Theta(\log \log n)$ with probability at least $1 - 1/n$.

This is called the power of two choices :
It is a standard technique to improve the performance of randomised algorithms.

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Randomised QuickSort

Chernoff Bound: Proof

Chernoff Bound

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[\,X\,] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[\,X \geq (1+\delta)\mu\,] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}.$$

Proof:

1. For $\lambda > 0$,

$$\mathbf{P}[\,X \geq (1+\delta)\mu\,] \underset{e^{\lambda x}\text{ is incr}}{\leq} \mathbf{P}\left[\,e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}\,\right] \underset{\text{Markov}}{\leq} e^{-\lambda(1+\delta)\mu}\mathbf{E}\left[\,e^{\lambda X}\,\right]$$

2. $\mathbf{E}\left[\,e^{\lambda X}\,\right] = \mathbf{E}\left[\,e^{\lambda \sum X_i}\,\right] \underset{\text{indep}}{=} \prod_{i=1}^{n} \mathbf{E}\left[\,e^{\lambda X_i}\,\right]$

3. $$\mathbf{E}\left[\,e^{\lambda X_i}\,\right] = e^{\lambda}p_i + (1-p_i) = 1 + p_i(e^{\lambda} - 1) \underset{\substack{1+x\leq e^x\\ \text{for } x>0}}{\leq} e^{p_i(e^{\lambda}-1)}$$

Chernoff Bound: Proof

1. For $\lambda > 0$,

$$\mathbf{P}[\,X \geq (1+\delta)\mu\,] \underset{e^{\lambda x} \text{ is incr}}{=} \mathbf{P}\left[\,e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}\,\right] \underset{\text{Markov}}{\leq} e^{-\lambda(1+\delta)\mu}\mathbf{E}\left[\,e^{\lambda X}\,\right]$$

2. $\mathbf{E}[\,e^{\lambda X}\,] = \mathbf{E}\left[\,e^{\lambda \sum X_i}\,\right] \underset{\text{indep}}{=} \prod_{i=1}^{n} \mathbf{E}[\,e^{\lambda X_i}\,]$

3. $$\mathbf{E}\left[\,e^{\lambda X_i}\,\right] = e^{\lambda} p_i + (1-p_i) = 1 + p_i(e^{\lambda}-1) \underset{\substack{1+x \leq e^x \\ \text{for } x>0}}{\leq} e^{p_i(e^{\lambda}-1)}$$

4. Putting all together

$$\mathbf{P}[\,X \geq (1+\delta)\mu\,] \leq e^{-\lambda(1+\delta)\mu} \prod_{i=1}^{n} e^{p_i(e^{\lambda}-1)} = e^{-\lambda(1+\delta)\mu} e^{\mu(e^{\lambda}-1)}$$

5. Choose $\lambda = \log(1+\delta) > 0$ to get the result.

Chernoff Bound: The recipe

The proof of the Chernoff bound is based in three key steps. These are

1. Let $\lambda > 0$, then

$$\mathbf{P}[\, X \geq (1+\delta)\mu \,] \leq e^{-\lambda(1+\delta)\mu} \mathbf{E}\left[\, e^{\lambda X} \,\right]$$

2. Compute an upper bound for $\mathbf{E}[\, e^{\lambda X} \,]$ **(This is the hard one)**
3. Optimise the value of $\lambda > 0$.

The function $\lambda \to \mathbf{E}[\, e^{\lambda X} \,]$ is called the **moment-generating function** of X and it is very important to obtain sharp concentration inequalities.

Exercise: prove that $\mathbf{P}[\, X \geq t \,] \leq e^{-\mu} \left(\frac{e\mu}{t}\right)^t$,

Chernoff Bounds: Lower Tails

We can also use Chernoff Bounds to shows that a random variable is **not too small** compared to its mean.

Chernoff Bounds: Lower Tails

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[\,X\,] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[\,X \leq (1-\delta)\mu\,] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu}.$$

and for any $t < \mu$

$$\mathbf{P}[\,X \leq t\,] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t}.$$

Exercise: Prove it.

Hint: multiply both sides by -1 and repeat the proof of the Chernoff Bound

The useful Chernoff Bounds

Our current form of the Chernoff Bound is rather atrocious. We can derive a slightly weaker but more readable version.

Nicer Chernoff Bounds

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[\,X\,] = \sum p_i$. Then,

- For all $t > 0$,

$$\mathbf{P}[\,X \geq \mathbf{E}[\,X\,] + t\,] \leq e^{-2t^2/n}$$

$$\mathbf{P}[\,X \leq \mathbf{E}[\,X\,] - t\,] \leq e^{-2t^2/n}$$

- For $0 < \delta < 1$,

$$\mathbf{P}[\,X \geq (1+\delta)\mathbf{E}[\,X\,]\,] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[\,X\,]}{3}\right)$$

$$\mathbf{P}[\,X \leq (1-\delta)\mathbf{E}[\,X\,]\,] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[\,X\,]}{2}\right)$$

Exercise: Prove it

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Applications: QuickSort

Quick sort is a sorting algorithm that works as following.

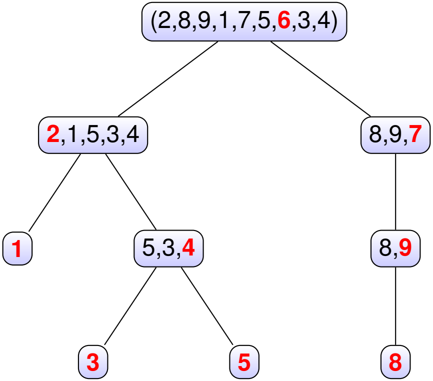
Algorithm: QuickSort

Input: Array of different number A.
Output: array A sorted in increasing order

- Pick an element from the array, the so-called pivot .
- If $|A| = 0$ or $|A| = 1$; return A.
- Else
  - Generate two subarrays A_1 and A_2:
    A_1 contains the elements that are smaller than the pivot ;
    A_2 contains the elements that are greater than the pivot ;
  - Recursively sort A_1 and A_2.

E.g. Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$, choose 6 as pivot, then we get $A_1 = (2, 1, 5, 3, 4)$ and $A_2 = (8, 9, 7)$.
It is well-known that the worst-case complexity (number of comparisons) of quick sort is $O(n^2)$. This happens when pivots are pretty bad, generating one large array and one small array.

Applications: QuickSort



Note that the number of comparison performed in quick sort is equivalent to the sum of the height of all nodes in the tree. In this case

$$0+1+1+2+2+2+3+3+3=17.$$

Applications: QuickSort

How to pick a good pivot ? we don't, **just pick one at random.**

This should be your standard answer in this course

Let's analyse quicksort with random pivots.

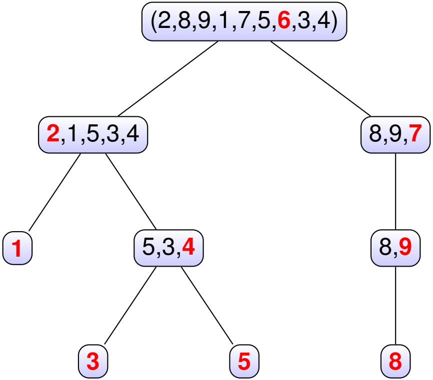
1. Consider n different number, wlog, $\{1, \ldots, n\}$
2. let H_i be the last level where i appears in the tree. Then the number of comparison is $H = \sum_{i=1}^{n} H_i$
3. we will prove that exists $C > 0$ such that

$$\mathbf{P}[\forall i, H_i \leq C \log n] \geq 1 - 1/n$$

4. actually, we will prove something equivalent but easier: we will prove that all leaves of the tree are at distance at most $C \log n$ from the root with probability at least $1 - 1/n$.
5. then $H = \sum_{i=1}^{n} H_i \leq Cn \log n$, with probability at least $1 - 1/n$.

Applications: QuickSort

- Let P be a path from the root to a leaf. A node in P is called **good** if the corresponding pivot partition the array into two subarrays each of size at least $1/3$ of the previous one, the node is **bad** otherwise.
- Denote by s_t the size of the array at level t in P.



E.g. Path: $(2,8,9,1,7,5,\mathbf{6},3,4) \to (\mathbf{2},1,5,3,4) \to (5,3,\mathbf{4}) \to (\mathbf{5})$

The vertices are: good, bad, good

$s_0 = 9, s_1 = 5, s_2 = 3, s_3 = 1.$

Applications: QuickSort

- Let P be a path from the root to a leaf. A node in P is called **good** if the corresponding pivot partition the array into two subarrays each of size at least $1/3$ of the previous one, the node is **bad** otherwise.
- Denote by s_t the size of the array at level t in P.
- After a good vertex we have that $s_t \leq (2/3)s_{t-1}$.
- Therefore, there are at most $T = \frac{\log n}{\log(3/2)} \leq 2\log n$ good nodes in a path P,
- Set $C = 21$ and suppose that $|P| > C\log n$.
- this implies that the number of bad vertices in the first $21\log n$ nodes is more than $19\log n$.

- Consider the first $\lfloor 21 \log n \rfloor$ vertices of P. Denote by $X_i = 1$ if the node at height i of P is bad, and $X_i = 0$ if it is good. Let $X = \sum_{i=1}^{\lfloor 21 \log n \rfloor} X_i$.
- Note that the X_i's are independent and $\mathbf{P}[\, X_i = 1\,] = 2/3$, and $\mathbf{E}[\, X\,] = (2/3)21 \log n = 14 \log n$. Then, by the (nicer) Chernoff Bounds

$$\mathbf{P}[\, X > \mathbf{E}[\, X\,] + t\,] \leq e^{-2t^2/n}$$

$$\begin{aligned} \mathbf{P}[\, X > 19 \log n\,] = \mathbf{P}[\, X > \mathbf{E}[\, X\,] + 5 \log n\,] &\leq e^{-2(5 \log n)^2/(21 \log n)} \\ &= e^{-(50/21) \log n} \leq 1/n^2. \end{aligned}$$

- Hence, we conclude the path has more than $21 \log n$ nodes with probability at most n^{-2}. There are at most n leaves, then by union bound , the probability that at least one path has more than $21 \log n$ nodes is n^{-1}

Applications: QuickSort

Remarks

- It is know that no sorting algorithm based on comparison takes less than $\Omega(n \log n)$
- The constant C can be improved a little bit, but in any case we will obtain that our randomised version of QuickSort that whp compares $O(n \log n)$ pairs
- It is possible to deterministically choose the best pivot that divide the array into two subarrays of the same size.
- The later requires to compute the median of the array in linear time, which is not easy to do
- Randomised solution for QuickSort is much easier to implement.