Lecture 4: Concentration Inequalities

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The useful Chernoff Bounds

Remember from last class.

Nicer Chernoff Bounds

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum p_i$. Then,

- For all $t > 0$,

$$\mathbf{P}[X \geq \mathbf{E}[X] + t] \leq e^{-2t^2/n}$$

$$\mathbf{P}[X \leq \mathbf{E}[X] - t] \leq e^{-2t^2/n}$$

- For $0 < \delta < 1$,

$$\mathbf{P}[X \geq (1+\delta)\mathbf{E}[X]] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[X]}{3}\right)$$

$$\mathbf{P}[X \leq (1-\delta)\mathbf{E}[X]] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[X]}{2}\right)$$

Outline

Randomised QuickSort

Extension of Chernoff Bounds

Examples

Applications: QuickSort

Quick sort is a sorting algorithm that works as following.

Algorithm: QuickSort

Input: Array of different number A.
Output: array A sorted in increasing order

- Pick an element from the array, the so-called pivot .
- If $|A| = 0$ or $|A| = 1$; return A.
- Else
  - Generate two subarrays A_1 and A_2:
    A_1 contains the elements that are smaller than the pivot ;
    A_2 contains the elements that are greater than the pivot ;
  - Recursively sort A_1 and A_2.

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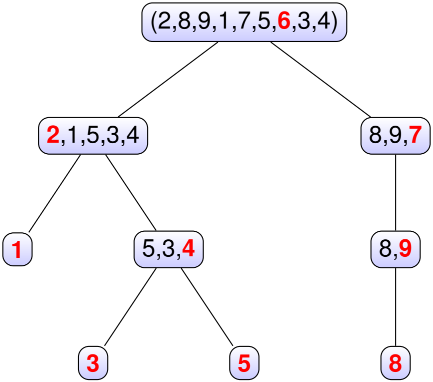
E.g. Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$, choose 6 as pivot, then we get $A_1 = (2, 1, 5, 3, 4)$ and $A_2 = (8, 9, 7)$.
It is well-known that the worst-case complexity (number of comparisons) of quick sort is $O(n^2)$. This happens when pivots are pretty bad, generating one large array and one small array.

Applications: QuickSort



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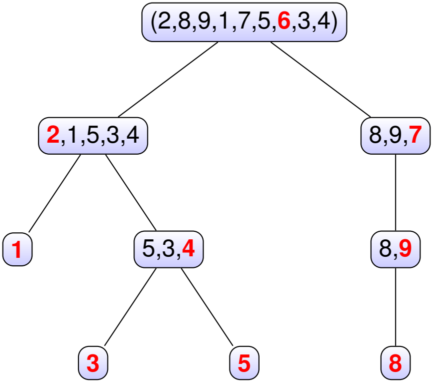

Note that the number of comparison performed in quick sort is equivalent to the sum of the height of all nodes in the tree. In this case

$$0 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 3 = 17.$$

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5. then $H = \sum_{i=1}^{n} H_i \leq Cn \log n$, with probability at least $1 - 1/n$.

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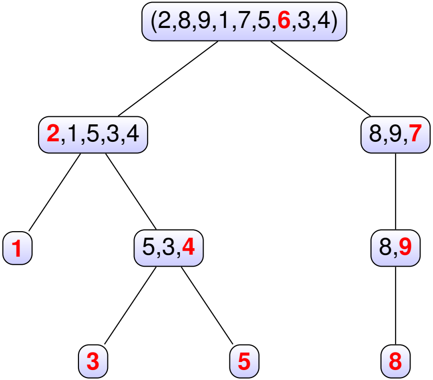

E.g. Path: $(2, 8, 9, 1, 7, 5, \mathbf{6}, 3, 4) \to (\mathbf{2}, 1, 5, 3, 4) \to (5, 3, \mathbf{4}) \to (\mathbf{5})$
The vertices are: good, bad, good
$s_0 = 9, s_1 = 5, s_2 = 3, s_3 = 1.$

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- Therefore, there are at most $T = \frac{\log n}{\log(3/2)} \leq 2 \log n$ good nodes in a path P,
- Set $C = 21$ and suppose that $|P| > C \log n$.
- this implies that the number of bad vertices in the first $21 \log n$ nodes is more than $19 \log n$.

- Consider the first $\lfloor 21 \log n \rfloor$ vertices of P. Denote by $X_i = 1$ if the node at height i of P is bad, and $X_i = 0$ if it is good. Let $X = \sum_{i=1}^{\lfloor 21 \log n \rfloor} X_i$.

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- Note that the X_i's are independent and $\mathbf{P}[\, X_i = 1\,] = 2/3$, and $\mathbf{E}[\, X\,] = (2/3)21 \log n = 14 \log n$. Then, by the (nicer) Chernoff Bounds

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- Hence, we conclude the path has more than $21 \log n$ nodes with probability at most n^{-2}. There are at most n leaves, then by union bound , the probability that at least one path has more than $21 \log n$ nodes is n^{-1}

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- It is possible to deterministically choose the best pivot that divide the array into two subarrays of the same size.
- The later requires to compute the median of the array in linear time, which is not easy to do
- Randomised solution for QuickSort is much easier to implement.

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Extension of Chernoff Bounds

Examples

Chernoff Bound: Extension to other Random Variables

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Remember the key steps:

Chernoff Bound recipe

1. Let $\lambda > 0$, then

$$\mathbf{P}[\, X \geq (1+\delta)\mu \,] \leq e^{-\lambda(1+\delta)\mu} \mathbf{E}\left[\, e^{\lambda X} \,\right]$$

2. Compute an upper bound for $\mathbf{E}\left[\, e^{\lambda X} \,\right]$
3. Optimise the value of $\lambda > 0$.

Exercise:

- Let X be a Poisson random variable of mean μ. Prove that

$$\mathbf{E}\left[e^{\lambda X} \right] = e^{\mu(e^{\lambda}-1)}$$

and deduce that for $t \geq \mu$

$$\mathbf{P}[\, X \geq t \,] \leq e^{-\mu} \left(\frac{e\lambda}{t} \right)^{t} \quad \text{and} \quad \mathbf{P}[\, X \geq (1+\delta)\mu \,] \leq e^{-\delta^2 \mu},$$

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- Let X be a Normal random variable of mean μ and variance σ^2. Prove that

$$\mathbf{E}\left[e^{\lambda X} \right] = e^{\mu\lambda+\sigma^2\lambda^2/2},$$

and deduce that for $t > \mu$

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Hoeffding's Extension Lemma

Let X be a random variable with mean 0 such that $a \leq X \leq b$, then for all $\lambda \in \mathbb{R}$.

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We will not study the proof of this Lemma

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This is not magic! you just need to optimise on λ

Chernoff-Bounds: Final Remarks

- There are several version of Chernoff-style Bounds that work for sum of independent random variables.
- The proof of all of them usually follows the same **recipe**
- Some bounds include more information about the random variables, e.g. the variance
- the limit is the amount of information we have about the random variables and our ability to manipulate/bound quantities.

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- but in general it is very hard to compute moment generating functions
- It is worth trying to transform the problem into the setting of sum of independent random variable
- There is one more very important bound

Method of Bounded Differences

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Some examples:

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We can simply prove concentration of X around it means by the so-called Method of Bounded Differences

Method of Bounded Differences

A function f is called Liptchitz of parameter $\mathbf{c} = (c_1, \ldots, c_n)$ if for all i

$$|f(x_1, x_2, \ldots, x_{i-1}, \boldsymbol{x_i}, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, \boldsymbol{y_i}, x_{i+1}, \ldots, x_n)| \leq c_i$$

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McDiarmid's inequality

Let $X_1, \ldots, X_n$ be independent random variables. Let f be Liptchitz of parameter $\mathbf{c} = (c_1, \ldots, c_n)$. Let $X = f(X_1, \ldots, X_n)$. Then

$$\mathbf{P}[\, X - \mathbf{E}[\, X \,] \geq t \,] \leq \exp\left(-\frac{2t^2}{\sum c_i^2}\right)$$

and

$$\mathbf{P}[\, X - \mathbf{E}[\, X \,] \leq -t \,] \leq \exp\left(-\frac{2t^2}{\sum c_i^2}\right)$$

We will not study the Proof of McDiarmid's Inequality

Outline

Randomised QuickSort

Extension of Chernoff Bounds

Examples

Examples: Balls into Bins

- Consider m balls assigned uniformly at random into n bins.
- Enumerate the balls from 1 to m. Ball i is assigned to a random bin X_i.
- Let Z be the number of empty bins (after assigning the balls)
- $Z = f(X_1, \ldots, X_m)$ and f is Liptchitz with $\mathbf{c} = (1, \ldots, 1)$ (because if we move one ball to another bin, the number of empty bins changes at most in 1)
- By the McDiarmid's inequality

$$\mathbf{P}[\,|F - \mathbf{E}[\,F\,]\,| > t\,] \leq 2e^{-2t^2/m}$$

Example: Bin Packing

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6. Therefore

$$\mathbf{P}[\, B - \mathbf{E}[\, B\,] \geq t\,]\,, \mathbf{P}[\, B - \mathbf{E}[\, B\,] \leq -t\,] \leq e^{-2t^2/n}.$$

A random distance problem

Consider an n by n square grid $\{0, 1, \ldots, n\}^2$, where each point is connected to each of its (at most) four neighbours (N, S, E, W). Within each inner square of the grid, we draw a diagonal from SW to NE with probability p.

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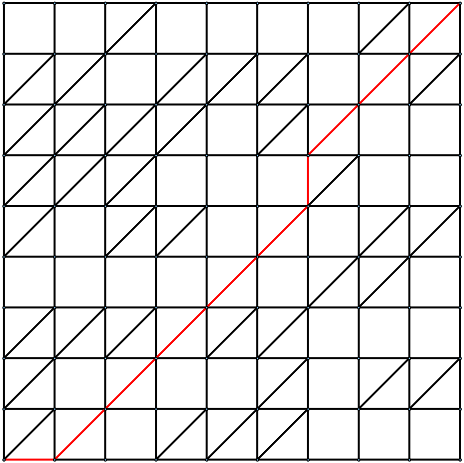

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   Note the second bound is way more useful than the first one.

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5. Therefore, for $t > 0$

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Observe this bound is better than the previous one

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6. C_S is Lipschitz with $\boldsymbol{c} = (1, \ldots, 1)$
7. Therefore, for $\delta > 0$,

$$\mathbf{P}[\, C_S - \mathbf{E}[\, C_S \,] \geq \delta \mathbf{E}[\, C_S \,] \,] \leq \exp\left(- \frac{2\delta^2 \mathbf{E}[\, C_S \,]^2}{|S|(n - |S|)} \right)$$

8. **Exercise:** Deduce that for any $S \subseteq [n]$,

$$\mathbf{P}\left[C_S \geq \frac{n^2}{8} + \delta \frac{n^2}{4} \right] \leq e^{-\Omega(\delta^2 n^2)}$$

9. By the union bound, we have that

$$\mathbf{P}\left[\exists S : C_S \geq \frac{n^2}{8} + \delta \frac{n^2}{4} \right] \leq 2^n e^{-\Omega(\delta^2 n^2)} = 2^n e^{-\Omega(c^2 n)}$$

10. Recall that $\delta = c/\sqrt{n}$, now we pick c to be large enough, such that $2^n e^{-\Omega(c^2 n)} = 2^{-n}$
11. The main result is:

Theorem

There is a constant c, such that w.h.p. the Max Cut in $G_{n,1/2}$ is at most $n^2/8 + cn^{3/2}$