Lecture 11: Graph clustering and random walks

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UNIVERSITY OF CAMBRIDGE

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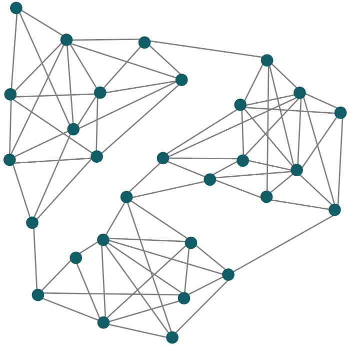
Graph clustering

Partition the graph into pieces (clusters) so that vertices in the same piece have, on average, more connections among each other than with vertices in other clusters



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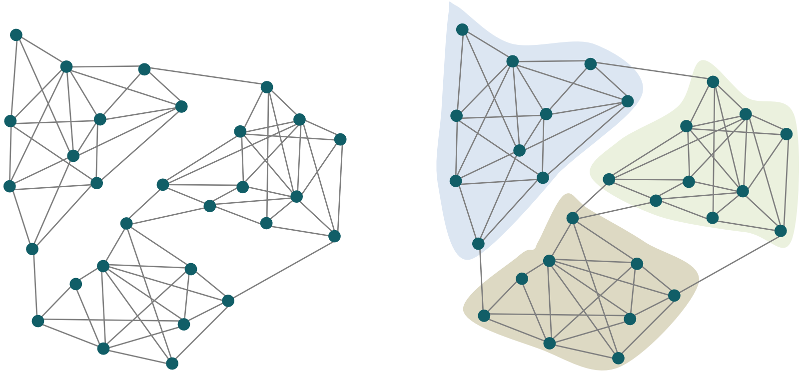

Why study graph clustering?

- Many practical applications, e.g.:
  - Community detection
  - Group webpages according to their topics
  - Find proteins performing the same function within a cell
  - Image segmentation
  - Identify bottlenecks in a network

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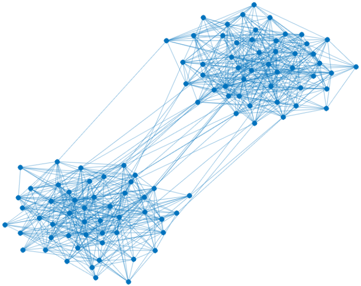
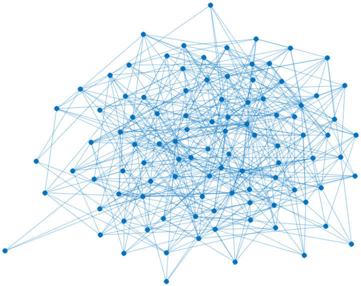
- Connections with different areas of mathematics and TCS, e.g.:
  - Random walk theory
  - Combinatorics
  - Theory of metric spaces
  - Approximation algorithms
  - Complexity theory

Relation between clustering and mixing

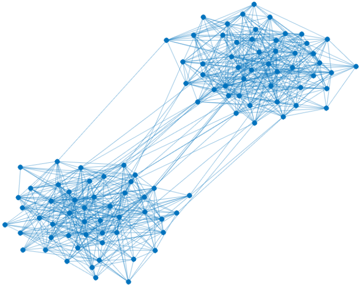
- Which graph has a "cluster-structure"?

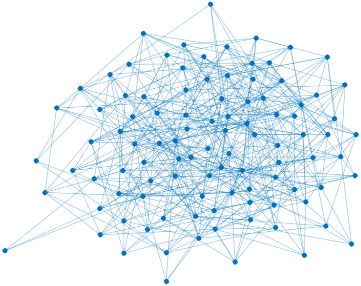




Relation between clustering and mixing

- Which graph has a “cluster-structure”?
- Which graph mixes faster?





Weighted graphs and random walks

$G = (V, E, w)$ with weight function w, s.t.

- $w\colon V \times V \to \mathbb{R}_{\geq 0}$
- $w(x, y) > 0 \iff \{x, y\} \in E$
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The transition matrix of a lazy random walk on G is the n by n matrix P defined as

$$P(x, y) = \frac{w(x, y)}{2d(x)}, \qquad P(x, x) = \frac{1}{2}$$

where $d(x) = \sum_{z \in V} w(x, z)$.
It has stationary distribution π s.t. $\pi(x) = \frac{d(x)}{\sum_z d(z)}$.

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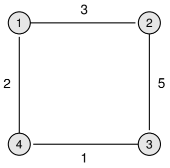

$$P = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} & 0 & \frac{1}{5} \\ \frac{3}{16} & \frac{1}{2} & \frac{5}{16} & 0 \\ 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{12} \\ \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$$

How do we formalise the concept of cluster/bottleneck?

Enter the conductance

Let $G = (V, E, w)$ and $\emptyset \neq S \subset V$.
The conductance (edge expansion) of S is

$$\phi(S) := \frac{w(S, V \setminus S)}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}}$$

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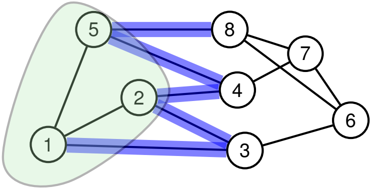

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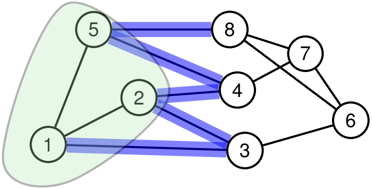

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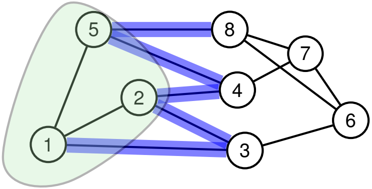

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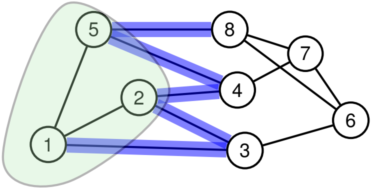

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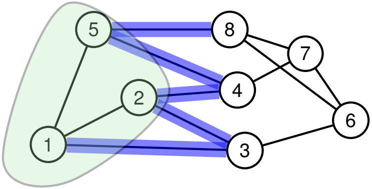
NP-hard to compute!



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- no constant factor approximation (in the worst case)
- mixing on G is $t_{mix} = O(\log(n)/\phi(G)^2)$.

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$

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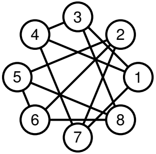

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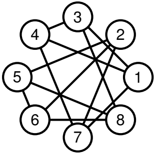

$$1 - \lambda_2 \approx 0.13$$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

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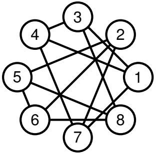

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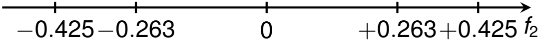
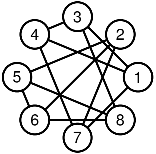


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

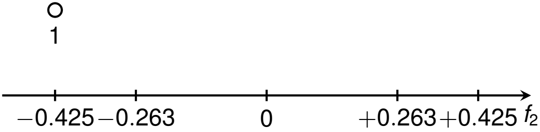
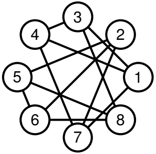


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$

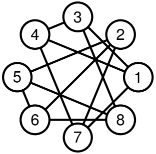


$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$

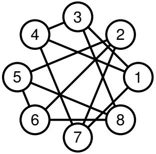


$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

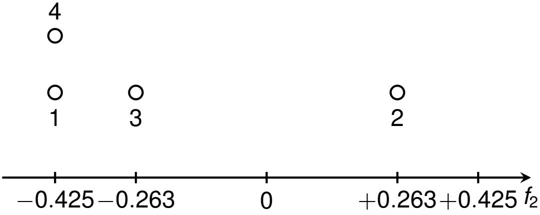
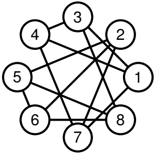


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

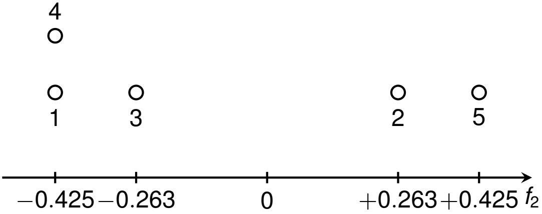
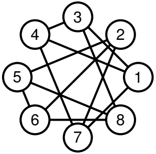


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0&0&1&1&0&0&1&0\\ 0&0&0&0&1&1&1&0\\ 1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&1&0\\ 0&1&0&0&0&1&0&1\\ 0&1&0&0&1&0&0&1\\ 1&1&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac12&0&\frac16&\frac16&0&0&\frac16&0\\ 0&\frac12&0&0&\frac16&\frac16&\frac16&0\\ \frac16&0&\frac12&\frac16&0&0&0&\frac16\\ \frac16&0&\frac16&\frac12&0&0&\frac16&0\\ 0&\frac16&0&0&\frac12&\frac16&0&\frac16\\ 0&\frac16&0&0&\frac16&\frac12&0&\frac16\\ \frac16&\frac16&0&\frac16&0&0&\frac12&0\\ 0&0&\frac16&0&\frac16&\frac16&0&\frac12 \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

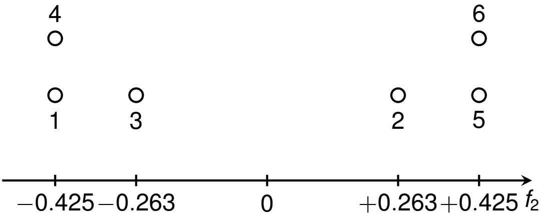
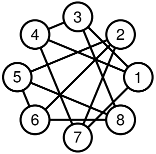


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

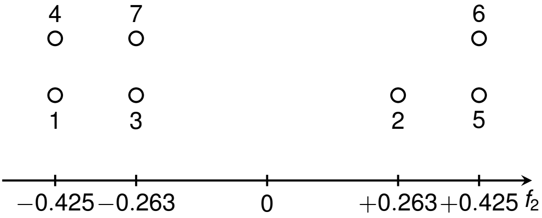
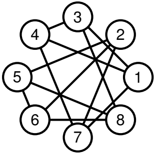


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0&0&1&1&0&0&1&0\\ 0&0&0&0&1&1&1&0\\ 1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&1&0\\ 0&1&0&0&0&1&0&1\\ 0&1&0&0&1&0&0&1\\ 1&1&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac12&0&\frac16&\frac16&0&0&\frac16&0\\ 0&\frac12&0&0&\frac16&\frac16&\frac16&0\\ \frac16&0&\frac12&\frac16&0&0&0&\frac16\\ \frac16&0&\frac16&\frac12&0&0&\frac16&0\\ 0&\frac16&0&0&\frac12&\frac16&0&\frac16\\ 0&\frac16&0&0&\frac16&\frac12&0&\frac16\\ \frac16&\frac16&0&\frac16&0&0&\frac12&0\\ 0&0&\frac16&0&\frac16&\frac16&0&\frac12 \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

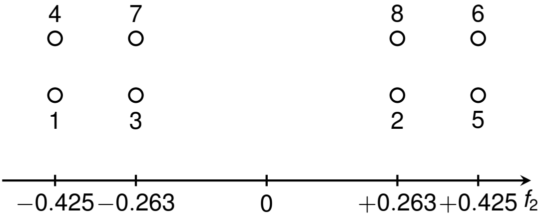
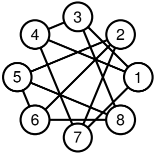


Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

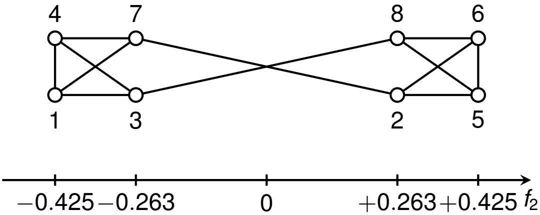
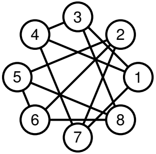


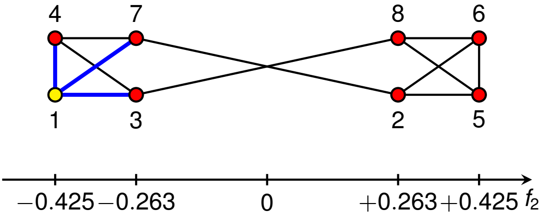
Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$$1 - \lambda_2 \approx 0.13$$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

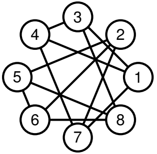


Sweep: 1

Conductance: 1

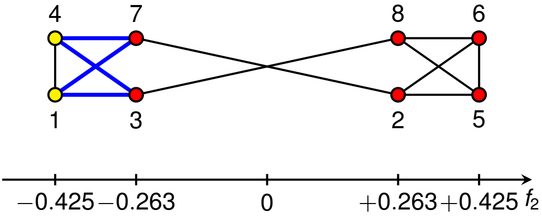
Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

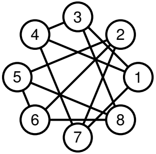


Sweep: 2

Conductance: 0.666

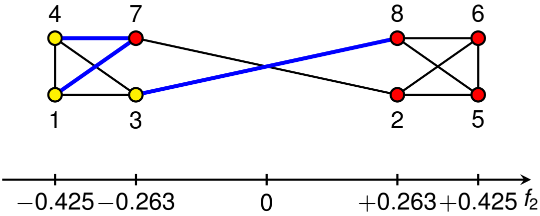
Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$$1 - \lambda_2 \approx 0.13$$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

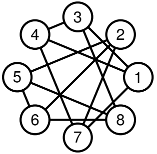


Sweep: 3

Conductance: 0.333

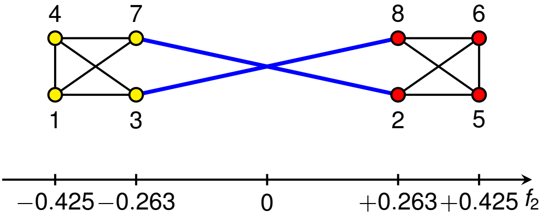
Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



$1 - \lambda_2 \approx 0.13$

$$f_2 = (-0.425, +0.263, -0.263, -0.425, +0.425, +0.425, -0.263, +0.263)^T$$

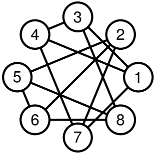


Sweep: 4

Conductance: 0.166

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



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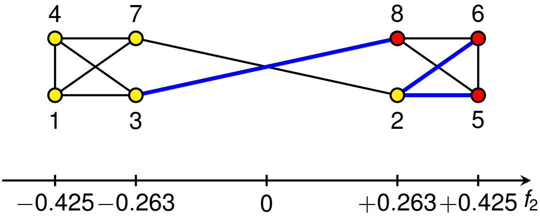
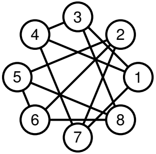


Sweep: 5

Conductance: 0.333

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



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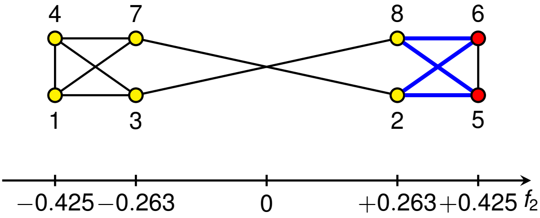
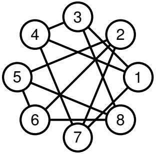


Sweep: 6

Conductance: 0.666

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



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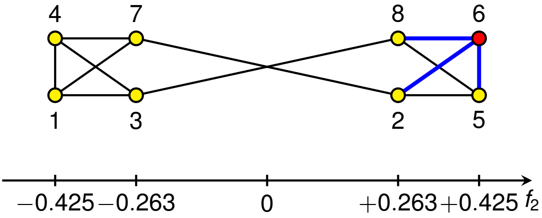
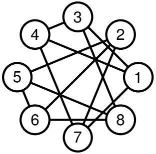


Sweep: 7

Conductance: 1

Illustration on a (very) small example

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \end{pmatrix}$$



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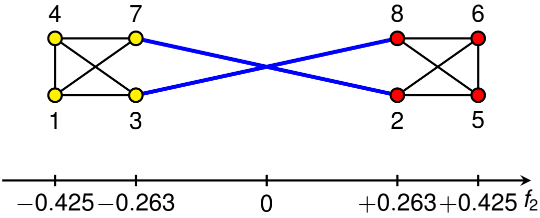

Best sweep: 4

Conductance: 0.166

Intuition

Let's start with a simplified example: $G = (V_1 \cup V_2, E)$

- disconnected with connected components supported on V_1 and V_2
- $|V_1| = |V_2| = n$
- regular (i.e., all vertices have the same degree)

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We want to find $f\colon V \to \mathbb{R}$ such that $f \perp 1$ minimises

$$1 - \lambda_2 = \min_{\substack{f \in \mathbb{R}^n \setminus \{0\} \\ f \perp 1}} \frac{\sum_{\{u,v\} \in E} (f(u) - f(v))^2}{2d \sum_{u \in V} f(u)^2}$$

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Hope: If $\phi(G)$ is small, a similar construction can give us a small spectral gap

Proof of the "easy" direction $(1 - \lambda_2)/2 \leq \phi(G)$

We prove it for *G d*-regular.

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We prove it for G d-regular.

Proof: Recall that $1 - \lambda_2 = \min_{\substack{f \in \mathbb{R}^n \setminus \{0\} \\ f \perp 1}} \frac{\sum_{\{u,v\} \in E} (f(u) - f(v))^2}{2d \sum_{u \in V} f(u)^2}$

- Take $S \subset V$ minimising $\phi(G)$

$\square$

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- Construct $f \in \mathbb{R}^n$ s.t. $f(u) = \begin{cases} 1/|S| & \text{if } u \in S \\ -1/|V \setminus S| & \text{if } u \notin S. \end{cases}$

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□

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- $\sum_{\{u,v\}\in E}(f(u)-f(v))^2 \leq \sum_{\substack{\{u,v\}\in E\\ u\in S, v\notin S}}\frac{4}{|S|^2} = \frac{4|E(S,V\setminus S)|}{|S|^2}$

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- $\sum_{\{u,v\}\in E} (f(u)-f(v))^2 \le \sum_{\substack{\{u,v\}\in E\\ u\in S, v\notin S}} \frac{4}{|S|^2} = \frac{4|E(S,V\setminus S)|}{|S|^2}$
- $1-\lambda_2 \le \frac{4|E(S,V\setminus S)|}{2d|S|^2} \cdot \frac{1}{1/|S|} = 2\phi(S) = 2\phi(G).$

□

Spectral partitioning example