COMPUTER SCIENCE TRIPOS Part II

Friday 15th March 2019 9:00 to 10:30

Module PC – Probability and Computation

Candidates should attempt all ***three*** *questions.*

Submit the answers on the paper provided.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

STATIONERY REQUIREMENTS
Script paper

SPECIAL REQUIREMENTS
Approved calculator permitted

SECTION A

1 (*a*) What is the definition of conductance for an undirected, unweighted graph $G = (V, E)$? [3 marks]

(*b*) Compute the conductance of the cycle with n vertices. [6 marks]

(*c*) What does part (*b*) imply for the mixing time of the cycle? [5 marks]

(*d*) Let P be a transition matrix of a Markov chain with state space Ω. Further, let μ and ν be two probability distributions on Ω. Prove that

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV} \,.$$

[6 marks]

2 Let $X_1, \ldots, X_n$ be independent random variables taking values in $[0, 1]$ with $\mathbf{E}[\, X_i \,] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $p = \sum_{i=1}^{n} p_i$.

(*a*) Prove that

$$\mathbf{E}\left[\, e^{\lambda X_i} \,\right] \leq p_i e^{\lambda} + (1 - p_i).$$

[6 marks]

(*b*) Prove that the following holds for any $\delta > 0$,

$$\mathbf{P}[\, X \geq (1+\delta)\mathbf{E}[\, X \,] \,] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mathbf{E}[X]} .$$

[8 marks]

[*Hint:* remember that $1 + x \leq e^x$ for each $x \geq 0$.]

Let $\{X_i\}_{i=0}^{\infty}$ be a sequence of independent random variables with $\mathbf{P}[\, X_i = 1 \,] = p$ and $\mathbf{P}[\, X_i = -1 \,] = q = 1 - p$ for each $i \geq 0$.

(*c*) Let $S_t = \sum_{i=0}^{t} X_i$ and suppose that $p \in (0, 1)$. Show that $M_t = (q/p)^{S_t}$ is a martingale with respect to $X_1, X_2, \ldots$. [5 marks]

(*d*) Let λ be a real number satisfying $0 < \lambda < 1$. Show that for any such λ there is some $C > 0$ such that $Z_t = C^t \lambda^{S_t}$ is a martingale with respect to $X_1, X_2, \ldots$. [6 marks]

3 For any integer $2 \le k \le n$, consider the problem of assigning numbers in $\{1, \ldots, k\}$ to the vertices of an n-vertex graph $G = (V, E)$. For every vertex $v \in V$, let $x_v \in \{1, \ldots, k\}$ be the number assigned to v. The objective is to maximise

$$C_x = \sum_{\{u,v\} \in E} \mathbf{1}_{x_u \neq x_v}.$$

Note that this is a generalisation of the MAX-CUT problem.

(a) Design a randomised algorithm which returns a solution satisfying

$$\mathbf{E}[\, C_x \,] \ge \left(1 - \frac{1}{k}\right) |E|.$$

[8 marks]

(b) Modify the algorithm so that, for any given $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$, the returned solution satisfies $C_x \ge \left(1 - \frac{1+\epsilon}{k}\right) |E|$ with probability at least $1 - \delta$. State explicitly the running time of your algorithm. [7 marks]

END OF PAPER