1 (*a*) What is the definition of conductance for an undirected, unweighted graph $G = (V, E)$? [3 marks]

(*b*) Compute the conductance of the cycle with n vertices. [6 marks]

(*c*) What does part (*b*) imply for the mixing time of the cycle? [5 marks]

(*d*) Let P be a transition matrix of a Markov chain with state space Ω. Further, let μ and ν be two probability distributions on Ω. Prove that

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV} .$$

[6 marks]

Answer:

(*a*) For any $v \in V$, let $d(v) = |\{u \colon \{u, v\} \in E\}|$. We define the volume of a set $S \subseteq V$ as $\mathrm{vol}(S) = \sum_{u \in S} d(u)$. Then, the conductance of G can be defined as

$$\phi(G) = \min_{\substack{\emptyset \neq S \subset V \\ \mathrm{vol}(S) \leq \mathrm{vol}(G)/2}} \frac{E(S, V \setminus S)}{\mathrm{vol}(S)}$$

where $E(S, V \setminus S) = |\{\{u, v\} \in E \colon u \in S, v \notin S\}|$.

(*b*) First of all, any non-empty set $S \subset V$ has $E(S, V \setminus S) \geq 2$ (equality holds if and only if the set is connected). Moreover, $\mathrm{vol}(S) = 2|S|$. Therefore,

$$\frac{E(S, V \setminus S)}{\mathrm{vol}(S)} = \frac{2}{2|S|} = \frac{1}{|S|}.$$

Therefore, $\phi(G) = \frac{2}{n}$ when n is even, or $\phi(G) = \frac{2}{n-1}$ when n is odd.

(*c*) By Cheeger's inequality, the spectral gap of the transition matrix of a lazy random walk on a graph G satisfies $(1 - \lambda)/2 \leq \phi(G) = O(\sqrt{1 - \lambda})$. Moreover, for a graph of n vertices, we have that $t_{mix} = O(\log(n)/(1 - \lambda))$. Therefore, the mixing time on the complete graph is $O(\log n)$, while the mixing time on the cycle is between $\Omega(n)$ and $O(n^2 \log(n))$.

(*d*) By the definition of total variation distance,

$$\begin{aligned}
\|\mu P - \nu P\|_{TV} = \|(\mu - \nu) P\|_{TV} &= \frac{1}{2} \sum_{x \in \Omega} \left| \sum_{y \in \Omega} (\mu(y) - \nu(y)) P(y, x) \right| \\
&\leq \frac{1}{2} \left| \sum_{y \in \Omega} |\mu(y) - \nu(y)| \sum_{x \in \Omega} P(y, x) \right| \\
&= \|\mu - \nu\|_{TV} ,
\end{aligned}$$

where in the last equality we have used the fact that $\sum_{x \in \Omega} P(y, x) = 1$.

---

2 Let $X_1, \ldots, X_n$ be independent random variables taking values in $[0,1]$ with $\mathbf{E}[\,X_i\,] = p_i$. Let $X = \sum_{i=1}^n X_i$ and $p = \sum_{i=1}^n p_i$.

(a) Prove that

$$\mathbf{E}\left[\,e^{\lambda X_i}\,\right] \le p_i e^{\lambda} + (1 - p_i).$$

[6 marks]

(b) Prove that the following holds for any $\delta > 0$,

$$\mathbf{P}[\,X \ge (1+\delta)\mathbf{E}[\,X\,]\,] \le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mathbf{E}[X]}.$$

[8 marks]

[*Hint:* remember that $1 + x \le e^x$ for each $x \ge 0$.]

Let $\{X_i\}_{i=0}^{\infty}$ be a sequence of independent random variables with $\mathbf{P}[\,X_i = 1\,] = p$ and $\mathbf{P}[\,X_i = -1\,] = q = 1 - p$ for each $i \ge 0$.

(c) Let $S_t = \sum_{i=0}^t X_i$ and suppose that $p \in (0,1)$. Show that $M_t = (q/p)^{S_t}$ is a martingale with respect to $X_1, X_2, \ldots$. [5 marks]

(d) Let λ be a real number satisfying $0 < \lambda < 1$. Show that for any such λ there is some $C > 0$ such that $Z_t = C^t \lambda^{S_t}$ is a martingale with respect to $X_1, X_2, \ldots$. [6 marks]

Answer:

(a) As was mentioned in the course (Lecture 6 slide 10), $f(x) = e^{\lambda x}$ is a convex function. Hence, the line segment from $(0, f(0))$ and $(1, f(1))$ is above the graph of $e^{\lambda x}$ in $[0,1]$.

Observe that $f(0) = 1$, $f(1) = e^{\lambda}$ and the line through $(0,1)$ and $(1, e^{\lambda})$ is given by

$$y(x) = (e^{\lambda} - 1)x + 1.$$

As the random variable X_i takes values in $[0,1]$ we obtain the following by convexity

$$e^{\lambda x} \le (e^{\lambda} - 1)x + 1.$$

Taking the expectation both sides yields

$$\mathbf{E}\left[\,e^{\lambda X_i}\,\right] \le (e^{\lambda} - 1)\mathbf{E}[\,X_i\,] + 1 = (e^{\lambda} - 1)p_i + 1 = 1 + p_i(e^{\lambda} - 1).$$

(b) We now follow the next few steps of the recipe for Chernoff-type bounds (these are given in lecture 5, slides 19-20): Let $\lambda \ge 0$, then

$$\mathbf{P}[\,X \ge (1+\delta)\mathbf{E}[\,X\,]\,] \le \mathbf{P}\left[e^{\lambda X} \ge e^{(1+\delta)\mathbf{E}[\,X\,]}\right] \le \mathbf{E}\left[\,e^{\lambda X}\,\right] e^{-(1+\delta)\lambda\mathbf{E}[\,X\,]}.$$

Also since $X = \sum_{i=1}^{n} X_i$, X_i are independent and $\mathbf{E}[\, X\,] = p$ we have the following by part (a)

$$\mathbf{E}[\, e^{\lambda X}\,] = \prod_{i=1}^{n} \mathbf{E}[\, e^{\lambda X_i}\,] \le \prod_{i=1}^{n}[1 + p_i(e^{\lambda} - 1)] \le \prod_{i=1}^{n} e^{p_i(e^{\lambda}-1)} = \exp\left(\sum_{i=1}^{n} p_i(e^{\lambda} - 1)\right)$$
$$= \exp\left(p(e^{\lambda} - 1)\right).$$

Finally,

$$\mathbf{P}[\, X \ge (1+\delta)\mathbf{E}[\, X\,]\,] \le \exp\left(-(1+\delta)\lambda p + p(e^{\lambda} - 1)\right). \quad (1)$$

Here we can guess the right value of λ or we can minimise the quantity inside the exponential. We will do the second route. Define

$$f(\lambda) = -(1+\delta)\lambda p + p(e^{\lambda} - 1).$$

We solve $f'(\lambda) = 0$. Note that

$$f'(\lambda) = -(1+\delta)p + pe^{\lambda},$$

and so $f'(\lambda) = 0$ is solved for $\lambda = \log(1+\delta)$. Also $f''(\lambda) = p > 0$ indicating such a value is a local minimum. Substitute $\lambda = \log(1+\delta)$ into (1) to get the answer.

(*c*) We check the three properties of being a Martingale. Clearly M_t is a function of S_t which is in turn a function of $X_1, \ldots, X_t$. Also $-n \le S_n \le n$ therefore

$$|M_t| \le \max\left\{\frac{p}{q}, \frac{q}{p}\right\}^n,$$

meaning that $\mathbf{E}[\,|M_t|\,] \le \infty$. Finally, for $t \ge 1$

$$\mathbf{E}[\, M_{t+1}|X_1, \ldots, X_t\,] = \left(\frac{q}{p}\right)^{S_t+1} \cdot p + \left(\frac{q}{p}\right)^{S_t-1} \cdot q = \left(\frac{q}{p}\right)^{S_t} \left(\frac{q}{p} \cdot p + \frac{p}{q} \cdot q\right) = \left(\frac{q}{p}\right)^{S_t} = M_t.$$

(*d*) To be a martingale we need to satisfy the three properties. First Z_t is clearly a function of $X_1, \ldots, X_t$. To check whether $\mathbf{E}[\,|Z_t|\,] < \infty$ we need to be careful. For any fixed λ and $C := C(\lambda)$ we have

$$|Z_t| \le \left(C \cdot \frac{1}{\lambda}\right)^t < \infty,$$

recall that $0 < \lambda < 1$ and $-t \le S_t \le t$, which implies $\mathbf{E}[\,|Z_t|\,] < \infty$.

Finally, in a similar fashion to M_{t+1}, we have the following for $t \ge 1$

$$\mathbf{E}[\, Z_{t+1}|X_1, \ldots, X_t\,] = C^{t+1}\left(\lambda^{S_t+1} \cdot p + \lambda^{S_t-1} \cdot q\right) = C^t \lambda^{S_t} \cdot C\left(\lambda p + \frac{q}{\lambda}\right)$$
$$= Z_t \cdot C\left(\lambda p + \frac{q}{\lambda}\right).$$

Pick $C = \left(\lambda p + \frac{q}{\lambda}\right)^{-1}$, then $C\left(\lambda p + \frac{q}{\lambda}\right) = 1$ and thus Z_t is a martingale.

3 For any integer $2 \leq k \leq n$, consider the problem of assigning numbers in $\{1, \ldots, k\}$ to the vertices of an n-vertex graph $G = (V, E)$. For every vertex $v \in V$, let $x_v \in \{1, \ldots, k\}$ be the number assigned to v. The objective is to maximise

$$C_x = \sum_{\{u,v\} \in E} \mathbf{1}_{x_u \neq x_v}.$$

Note that this is a generalisation of the MAX-CUT problem.

(a) Design a randomised algorithm which returns a solution satisfying

$$\mathbf{E}[\, C_x \,] \geq \left(1 - \frac{1}{k}\right) |E|.$$

[8 marks]

(b) Modify the algorithm so that, for any given $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$, the returned solution satisfies $C_x \geq \left(1 - \frac{1+\epsilon}{k}\right) |E|$ with probability at least $1 - \delta$. State explicitly the running time of your algorithm. [7 marks]

Answer:

(a) Simply assign every x_u independently and uniformly at random from $\{1, \ldots, k\}$. Then,

$$\begin{aligned}
\mathbf{E}[\, C_x \,] &= \mathbf{E}\left[\sum_{\{u,v\} \in E} \mathbf{1}_{x_u \neq x_v} \right] \\
&= \sum_{\{u,v\} \in E} \mathbf{E}[\, \mathbf{1}_{x_u \neq x_v} \,] \\
&= \sum_{\{u,v\} \in E} \mathbf{P}[\, x_u \neq x_v \,] \\
&= \sum_{\{u,v\} \in E} \left(1 - \frac{1}{k}\right) \\
&= \left(1 - \frac{1}{k}\right) \cdot |E|.
\end{aligned}$$

(b) We define $Y := |E| - C_x$. Then by part (a),

$$\mathbf{E}[\, Y \,] = \frac{1}{k} |E|.$$

By Markov's inequality, it follows that

$$\mathbf{P}\left[Y \geq (1 + \epsilon) \frac{|E|}{k} \right] \leq \frac{1}{1 + \epsilon}.$$

Thus,

$$\mathbf{P}\left[C_x \leq |E| - (1 + \epsilon) \frac{|E|}{k} \right] \leq \frac{1}{1 + \epsilon}.$$

By running the algorithm $\lceil \log_{1+\epsilon} \delta \rceil$ times and returning the solution with largest value of C_x (which we call $\tilde{C}$), we conclude that with probability at least $1 - \left(\frac{1}{1+\epsilon}\right)^{\lceil \log_{1+\epsilon}(1/\delta) \rceil} \geq 1 - \delta$, the returned solution $\tilde{C}$ satisfies

$$\tilde{C} \geq |E| - (1+\epsilon)\frac{|E|}{k} = \left(1 - \frac{1+\epsilon}{k}\right) \cdot |E|.$$

For the running time we must iterate the original "random guessing" algorithm $\lceil \log_{1+\epsilon} \delta \rceil$ times checking the size of the cut each time. In each iteration we must assign each vertex a label at random to each vertex, thus $O(|V|)$ complexity, and calculate C_x for the labelling given in that iteration, which has $O(|E|)$ complexity. Thus the total runtime is $O\left((\log_{1+\epsilon} \delta) \cdot |E|\right)$.