9: Viterbi Algorithm for HMM Decoding

Machine Learning and Real-world Data

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Last session: estimating parameters of an HMM

- The dishonest casino, dice edition.
- Two hidden states: L (loaded dice), F (fair dice).
- Input: dual tape of state and observation (dice outcome) sequences X and O.

(s_0)	F	F	F	F	L	L	L	F	F	F	F	L	L	L	L	F	F	(s_f)
(k_0)	1	3	4	5	6	6	5	1	2	3	1	4	3	5	4	1	2	(k_f)

- You estimated transition and emission probabilities (A and B, Task 7).

This session: decoding

- Now we can only observe the numbers that are thrown but we don't know which dice is currently in use. (more realistic unless the croupier is a friend?)
- We want the HMM to find out when the fair dice was out, and when the loaded dice was out.
- We need to write a decoder. (Task 8)

Decoding: finding the most likely path

- Definition of decoding: Finding the most likely hidden state sequence X that explains the observation O given the HMM parameters $\mu = (A, B)$.

$$\begin{aligned}\hat{X} = &\operatorname*{argmax}_{X} P(X, O; \mu) \\ = &\operatorname*{argmax}_{X} P(O|X; B)P(X|; A) \\ = &\operatorname*{argmax}_{X_1 \ldots X_T} \prod_{t=1}^{T} P(O_t|X_t; B)P(X_t|X_{t-1}; A)\end{aligned}$$

- Number of possible state sequences X is O(N^T) ($N =$number of unique hidden states); too large for brute force search.

Viterbi is a Dynamic Programming Application

(Reminder from Algorithms course)
We can use Dynamic Programming if two conditions apply:

- Optimal substructure property
  - An optimal state sequence $X_1 \dots X_j \dots X_T$ contains inside it the sequence $X_1 \dots X_j$, which is also optimal
- Overlapping subsolutions property
  - If both X_t and X_u are on the optimal path, with $u > t$, then the calculation of the probability for being in state X_t is part of each of the many calculations for being in state X_u.

The intuition behind Viterbi

- Here's how we can save ourselves a lot of time.
- Because of the Limited Horizon of the HMM, we don't need to keep a complete record of how we arrived at a certain state.
  - For 1st-order HMM, we need to record one previous step.
- Just do the calculation of the probability of reaching each state once for each time step and memoise it in an appropriate data structure
  - This reduces our effort to $O(N^2T)$ for the 1st order HMM.
  - We need to calculate the probability of arriving in each hidden state given each previous hidden state for every timestep.
  - What if we had a 2nd order HMM?

Viterbi: main data structure

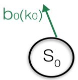
- Memoisation is done using a *trellis*.
- A trellis is equivalent to a Dynamic Programming table.
- The trellis is $(N+2) \times (T+2)$ in size, with states j as rows and time steps t as columns.
- Each cell j, t records the Viterbi probability $\delta_j(t)$, the probability of the most likely path that ends in state s_j at time t:
$$\delta_j(t) = \max_{1 \leq i \leq N} [\delta_i(t-1)\, a_{ij}\, b_j(O_t)]$$
- This probability is calculated by maximising over the best ways of going to s_j for each s_i.
- a_{ij}: the transition probability from s_i to s_j
- $b_j(O_t)$: the probability of emitting O_t from destination state s_j

Viterbi algorithm, initialisation

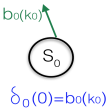
Note: the probability of a state starting the sequence at $t = 0$ is just the probability of it emitting the first symbol.

Viterbi algorithm, initialisation

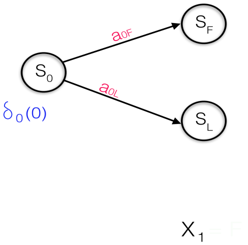
Viterbi algorithm, initialisation



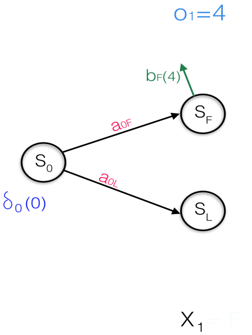
Viterbi algorithm, initialisation



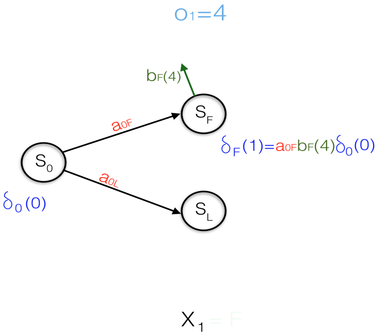
Viterbi algorithm, main step



Viterbi algorithm, main step: observation is 4



Viterbi algorithm, main step: observation is 4



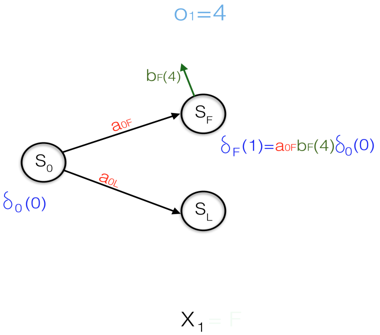
Viterbi algorithm, main step, ψ

- $\psi_j(t)$ is a helper variable that stores the $t-1$ state index i on the highest probability path.

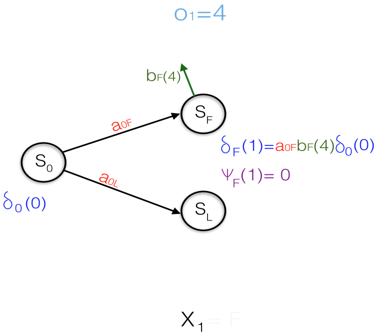
$$\psi_j(t) = \underset{1 \leq i \leq N}{\operatorname{argmax}}[\delta_i(t-1)\, a_{ij}\, b_j(O_t)]$$

- In the backtracing phase, we will use ψ to find the previous cell/state in the best path.

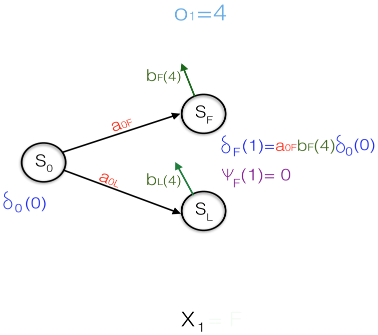
Viterbi algorithm, main step: observation is 4



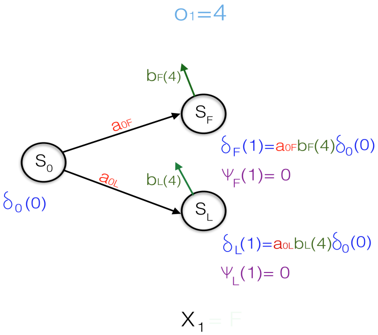
Viterbi algorithm, main step: observation is 4



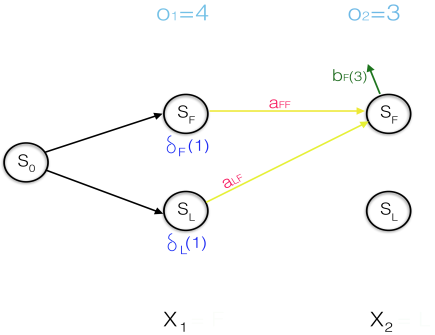
Viterbi algorithm, main step: observation is 4



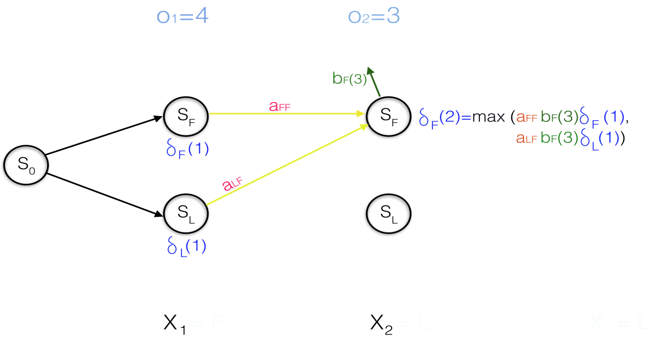
Viterbi algorithm, main step: observation is 4



Viterbi algorithm, main step: observation is 3



Viterbi algorithm, main step: observation is 3



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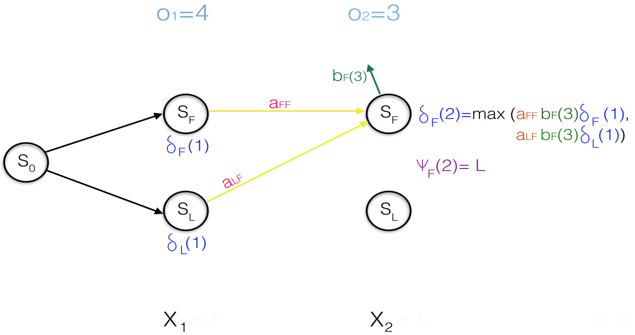

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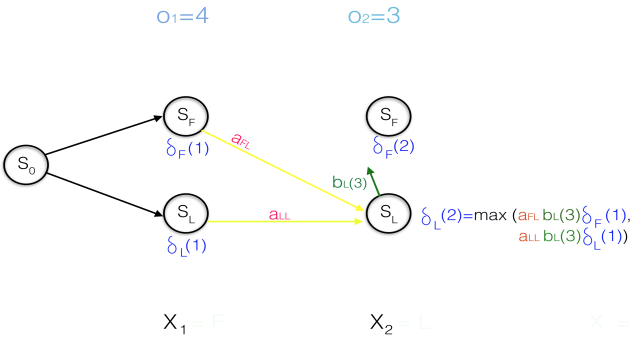
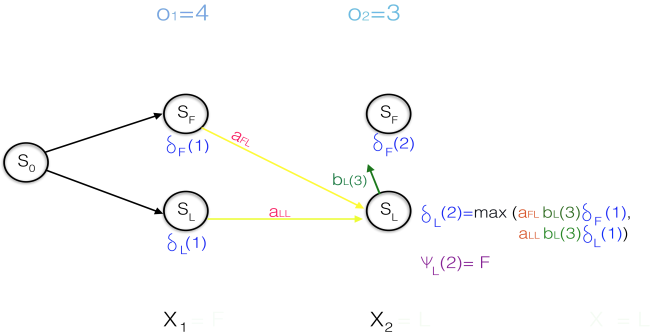
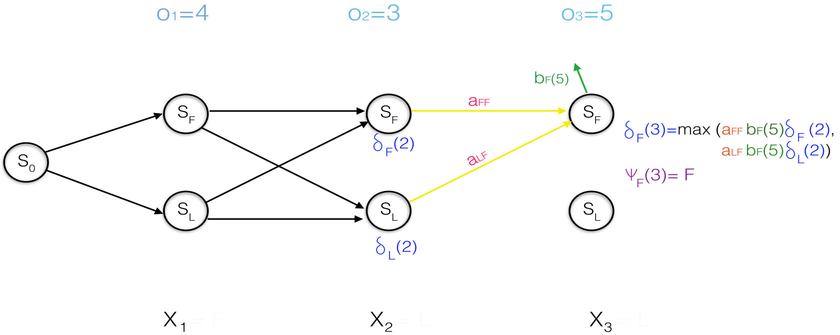


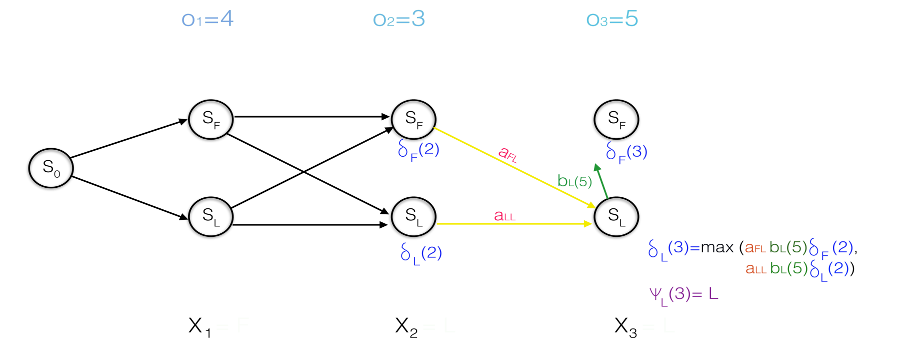
Viterbi algorithm, main step: observation is 3



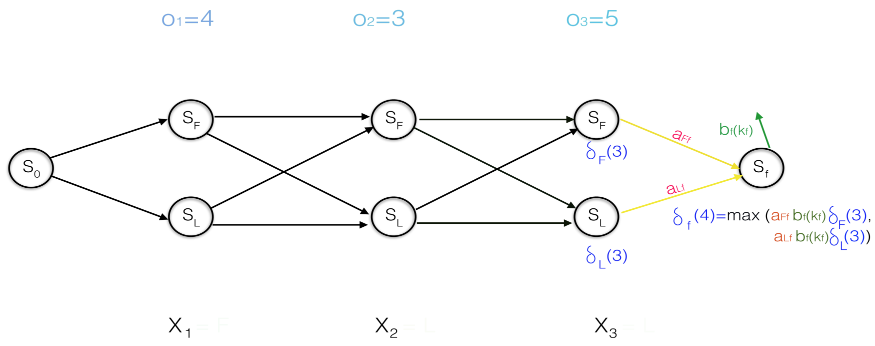
Viterbi algorithm, main step: observation is 5



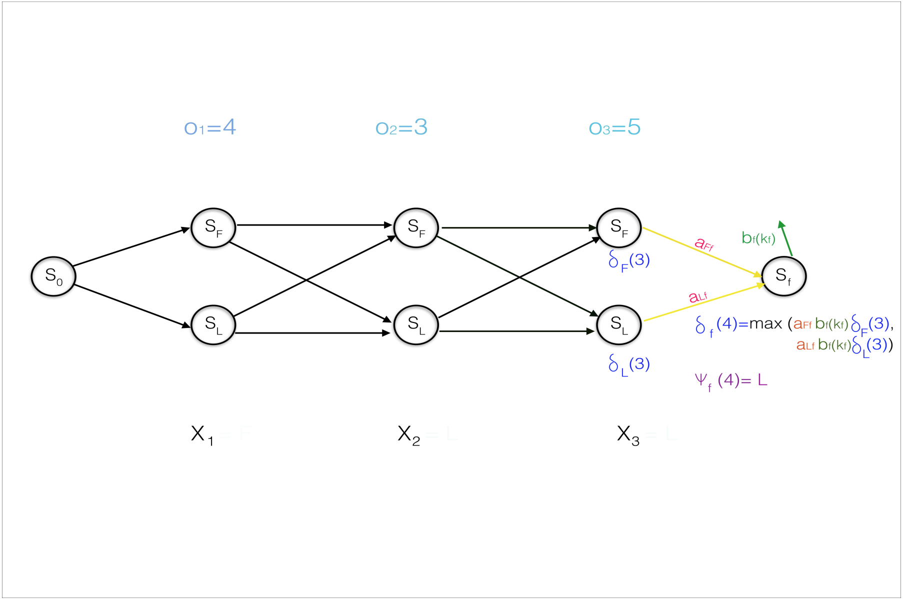
Viterbi algorithm, main step: observation is 5



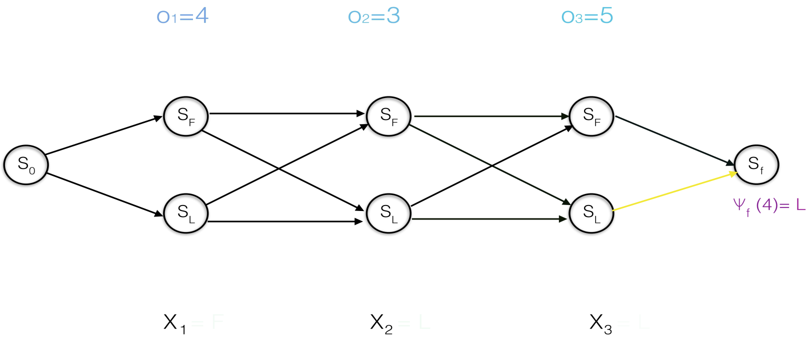
Viterbi algorithm, termination



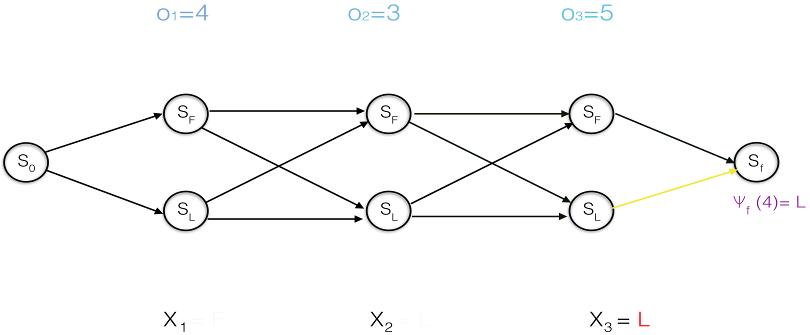
Viterbi algorithm, termination



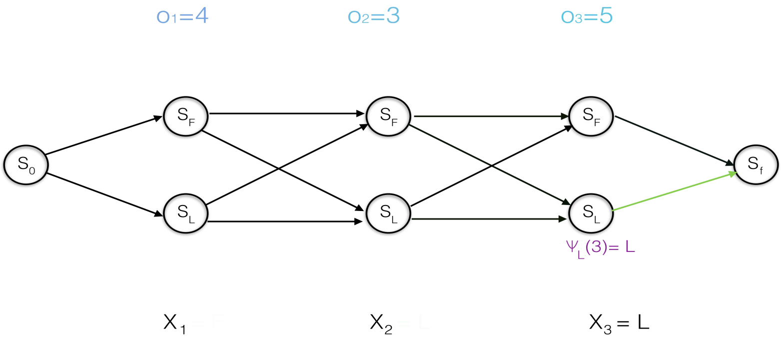
Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



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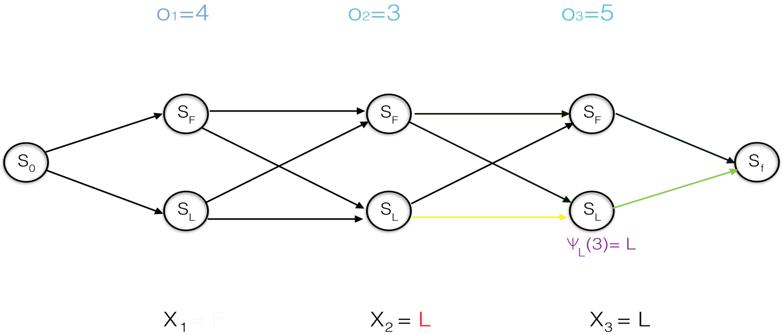

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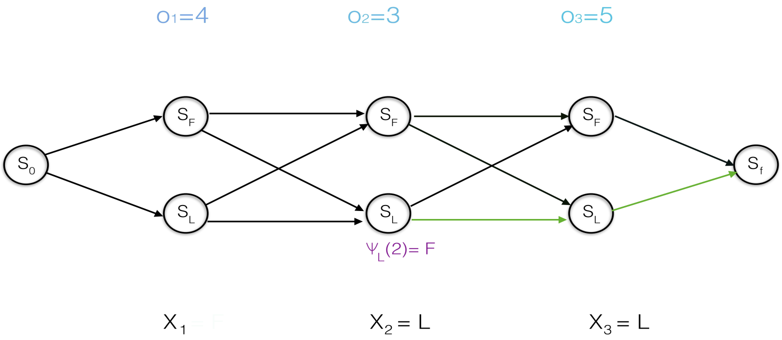
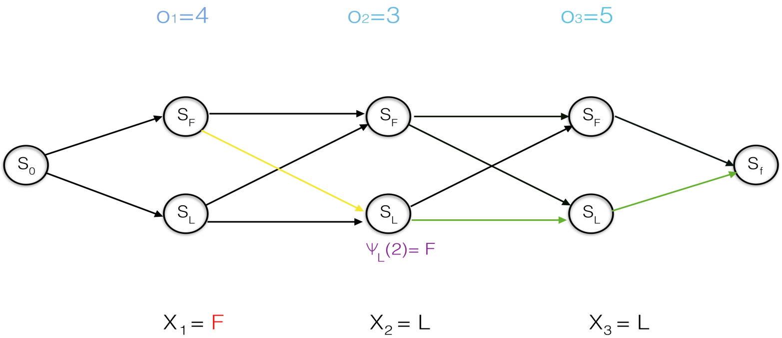
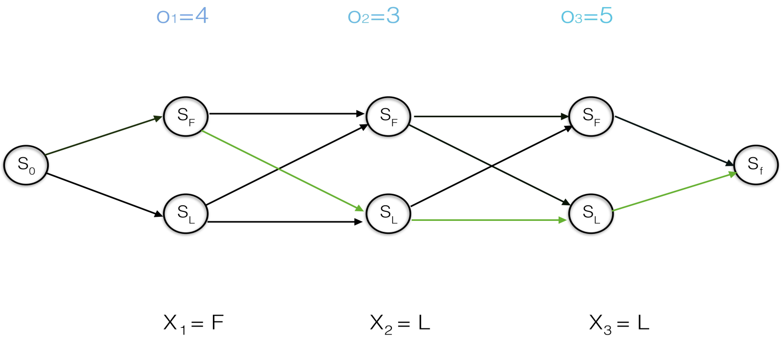


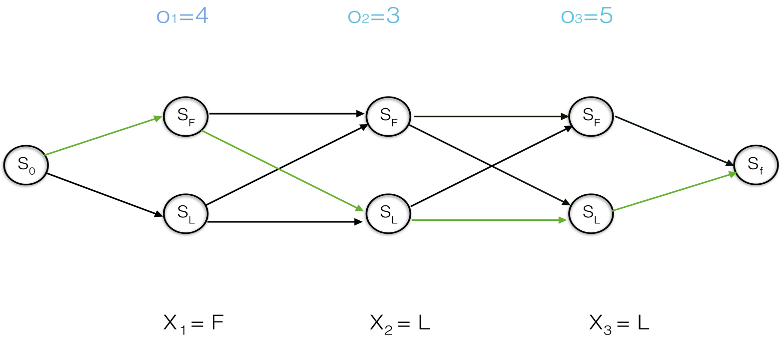
Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



Why is it necessary to keep N states at each time step?

- We have convinced ourselves that it's not necessary to keep more than N ("real") states per time step.
- But could we cut down the table to just a one-dimensional table of T time slots by choosing the probability of the best path overall ending in that time slot, in any of the states?
  - This would be the greedy choice
  - But think about what could happen in a later time slot.
  - You could encounter a zero or very low probability concerning all paths going through your chosen state s_j at time t.
  - Now a state s_k that looked suboptimal in comparison to s_j at time t becomes the best candidate.
  - As we don't know the future, we need to keep the probabilities for each state at each timestep.
- Viterbi is an exact decoding algorithm, find the same solution as brute-force but faster!

Precision and Recall

- So far, we have measured system success in accuracy.
- But sometimes it's only one type of instances that we find interesting.
- We don't want a summary measure that averages over interesting and non-interesting instances, as accuracy does.
- In those cases, we use precision, recall and F-measure.
- These metrics are imported from the field of information retrieval, where the difference beween interesting and non-interesting examples is particularly high.
- Accuracy doesn't work well when the types of instances are unbalanced

Precision and Recall

System says:

Truth is:		F	L	Total
	F	a	b	a+b
	L	c	d	c+d
	Total	a+c	b+d	a+b+c+d

- Precision of L: $P_L = \frac{d}{b+d}$
- Recall of L: $R_L = \frac{d}{c+d}$
- F-measure of L: $F_L = \frac{2P_L R_L}{P_L + R_L}$
- Accuracy: $A = \frac{a+d}{a+b+c+d}$

Your task today

Task 8:

- Implement the Viterbi algorithm.
- Run it on the dice dataset and measure precision of L (P_L), recall of L (R_L) and F-measure of L (F_L).

Literature

- Jurafsky and Martin, 3rd Edition, section 8.4 (but careful, notation!): `http://web.stanford.edu/~jurafsky/slp3/8.pdf`
- Fosler-Lussier, Eric (1998). Markov Models and Hidden Markov Models: A Brief Tutorial. TR-98-041.
- Smith, Noah A. (2011). Linguistic Structure Prediction (section 3.3.3)
- Bockmayr and Reinert (2011). Markov chains and Hidden Markov Models. Discrete Math for Bioinformatics WS 10/11.
- Extra reading on the connection between Viterbi and Dijkstra's algorithms (likely sequence vs shortest path): Liang Huang's tutorial: `https://www.aclweb.org/anthology/C08-5001.pdf`