13: Betweenness Centrality

Machine Learning and Real-world Data (MLRD)

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(based on slides created by Simone Teufel)

Lent 2020

Last session: some simple network statistics

- You measured the **degree** of each node and the **diameter** of the network.
- Next two sessions:
  - Today: finding **gatekeeper** nodes via **betweenness centrality**.
  - Next session: using betweenness centrality of edges to split graph into **cliques**.
- Reading for social networks (all sessions):
  - Easley and Kleinberg for background: Chapters 1, 2, 3 and first part of Chapter 20.
  - Brandes algorithm: two papers by Brandes (links in practical notes).

Intuition behind clique finding

- Certain nodes/edges are most crucial in linking densely connected regions of the graph: informally **gatekeepers.**
- Cutting those edges isolates the cliques/clusters.

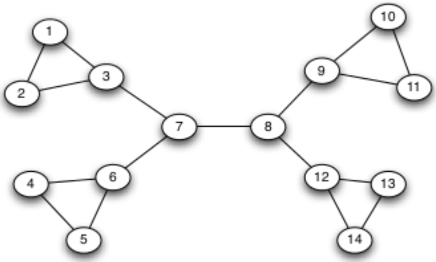


Figure 3-14a from Easley and Kleinberg (2010)

Intuition behind clique finding

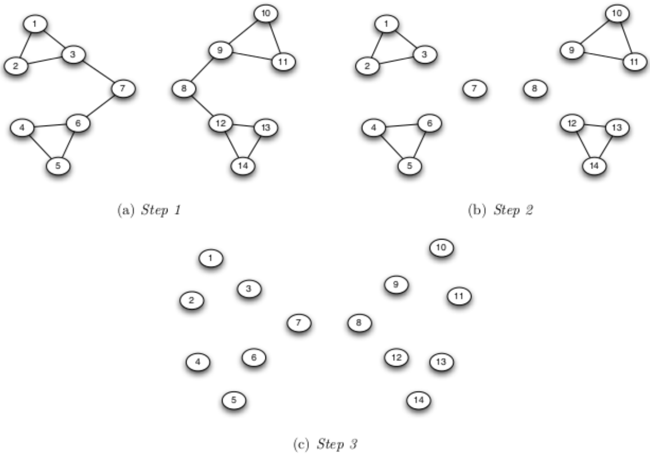


Figure 3-16 from Easley and Kleinberg (2010)

Gatekeepers: generalising the notion of local bridge

- Last time we saw the concept of **local bridge**: an edge which increased the shortest paths if cut.

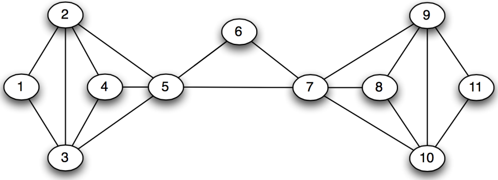
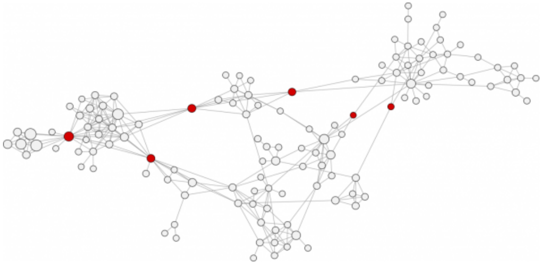


Figure 3-16 from Easley and Kleinberg (2010)

- But, more generally, the nodes that are intuitively the gatekeepers can be determined by **betweenness centrality**.

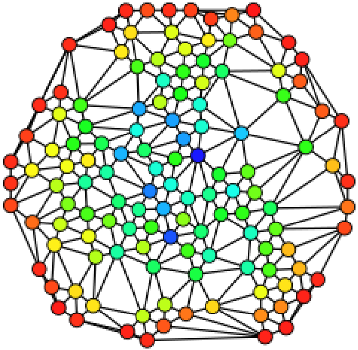
Betweenness centrality



https://www.linkedin.com/pulse/wtf-do-you-actually-know-who-influencers-walter-pike

- The betweenness centrality of a node V is defined in terms of the proportion of shortest paths that go through V for each pair of nodes.
- Here: the red nodes have high betweenness centrality.
- Note: Easley and Kleinberg talk about ‘flow’: misleading because we only care about shortest paths.

Betweenness, example



Claudio Rocchini: https://commons.wikimedia.org/wiki/File:Graph_betweenness.svg

- Betweenness: red is minimum; dark blue is maximum.

Betweenness centrality, formally (from Brandes 2008)

- Directed graph $G = < V, E >$
- $\sigma(s,t)$: number of shortest paths between nodes s and t
- $\sigma(s,t|v)$: number of shortest paths between nodes s and t that pass through v.
- $C_B(v)$, the betweenness centrality of v:

$$C_B(v) = \sum_{s,t \in V} \frac{\sigma(s,t|v)}{\sigma(s,t)}$$

- If $s = t$, then $\sigma(s,t) = 1$
- If $v \in s, t$, then $\sigma(s,t|v) = 0$

Number of shortest paths

- $\sigma(s,t)$ can be calculated recursively:

$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

  - Pred$(t) = \{u\colon (u,t) \in E, d(s,t) = d(s,u) + 1\}$ predecessors of t on shortest path from s
  - $d(s,u)$: Distance between nodes s and u
- This can be done by running Breadth First search with each node as source s once, for total complexity of $O(V(V+E))$.

Pairwise dependencies

- There are a cubic number of pairwise dependencies $\delta(s,t|v)$ where:
$$\delta(s,t|v) = \frac{\sigma(s,t|v)}{\sigma(s,t)}$$
- Naive algorithm uses lots of space.
- Brandes (2001) algorithm intuition: the dependencies can be aggregated without calculating them all explicitly.
- Recursive: can calculate dependency of s on v based on dependencies one step further away.

One-sided dependencies

Define **one-sided dependencies**:

$$\delta(s|v) = \sum_{t \in V} \delta(s,t|v)$$

Then Brandes (2001) shows:

$$\delta(s|v) = \sum_{\substack{(v,w) \in E \\ w:\, d(s,w) = d(s,v)+1}} \frac{\sigma(s,v)}{\sigma(s,w)}.(1 + \delta(s|w))$$

And:

$$C_B(v) = \sum_{s \in V} \delta(s|v)$$

Brandes algorithm

- Iterate over all vertices s in V
- Calculate $\delta(s|v)$ for all $v \in V$ in two phases:
  1. Breadth-first search, calculating distances and shortest path counts from s, push all vertices onto stack as they're visited.
  2. Visit all vertices in reverse order (pop off stack), aggregating dependencies according to equation.

Brandes (2008) pseudocode

Shortest-path vertex betweenness (Brandes, 2001).

input: directed graph $G = (V, E)$
data: queue Q, stack S (both initially empty)
  and for all $v \in V$:
  $dist[v]$: distance from source
  $Pred[v]$: list of predecessors on shortest paths from source
  $\sigma[v]$: number of shortest paths from source to $v \in V$
  $\delta[v]$: dependency of source on $v \in V$
output: betweenness $c_B[v]$ for all $v \in V$ (initialized to 0)

for $s \in V$ **do**
- ▼ **single-source shortest-paths problem**
  - ▼ **initialization**
    - **for** $w \in V$ **do** $Pred[w] \leftarrow$ empty list
    - **for** $t \in V$ **do** $dist[t] \leftarrow \infty; \quad \sigma[t] \leftarrow 0$
    - $dist[s] \leftarrow 0; \quad \sigma[s] \leftarrow 1$
    - enqueue $s \rightarrow Q$
  - **while** Q *not empty* **do**
    - dequeue $v \leftarrow Q$; push $v \rightarrow S$
    - **foreach** *vertex* w *such that* $(v, w) \in E$ **do**
      - ▼ **path discovery** // — w found for the first time?
        - **if** $dist[w] = \infty$ **then**
          - $dist[w] \leftarrow dist[v] + 1$
          - enqueue $w \rightarrow Q$
      - ▼ **path counting** // — edge (v, w) on a shortest path?
        - **if** $dist[w] = dist[v] + 1$ **then**
          - $\sigma[w] \leftarrow \sigma[w] + \sigma[v]$
          - append $v \rightarrow Pred[w]$
- ▼ **accumulation** // — back-propagation of dependencies
  - **for** $v \in V$ **do** $\delta[v] \leftarrow 0$
  - **while** S *not empty* **do**
    - pop $w \leftarrow S$
    - **for** $v \in Pred[w]$ **do** $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \cdot (1 + \delta[w])$
    - **if** $w \neq s$ **then** $c_B[w] \leftarrow c_B[w] + \delta[w]$

Step 1 - Prepare for BFS tree walk (Node A as s)

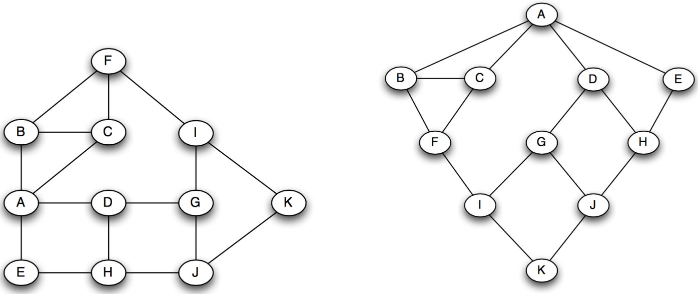
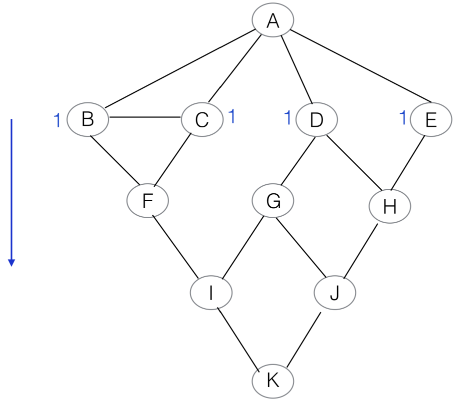


Figure 3-18 from Easley and Kleinberg (2010)

Brandes (2008) pseudocode: phase 1

```
while Q not empty do
    dequeue v ← Q;   push v → S
    foreach vertex w such that (v, w) ∈ E do
        ▼ path discovery   // — w found for the first time?
            if dist[w] = ∞ then
                dist[w] ← dist[v] + 1
                enqueue w → Q
        ▼ path counting   // — edge (v, w) on a shortest path?
            if dist[w] = dist[v] + 1 then
                σ[w] ← σ[w] + σ[v]
                append v → Pred[w]
```

Step 2 - Calculate $\sigma(s, v)$, the number of shortest paths between s and v



$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

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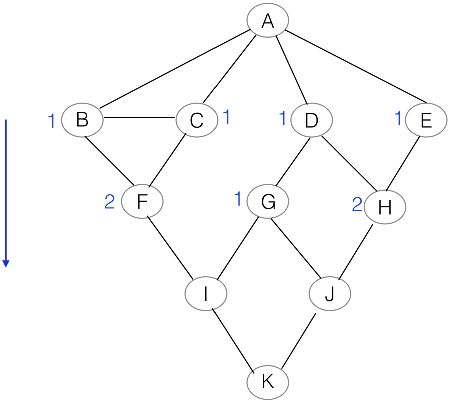

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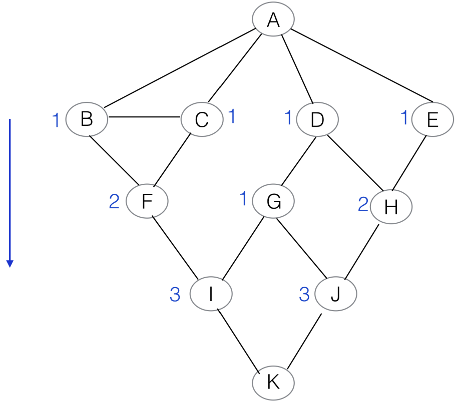

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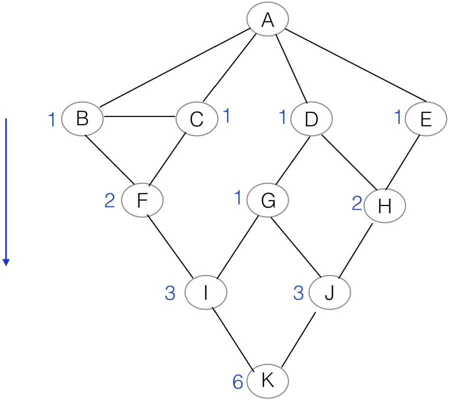


$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

Brandes (2008) pseudocode: phase 2

▼ **accumulation** // — back-propagation of dependencies

for $v \in V$ **do** $\delta[v] \leftarrow 0$

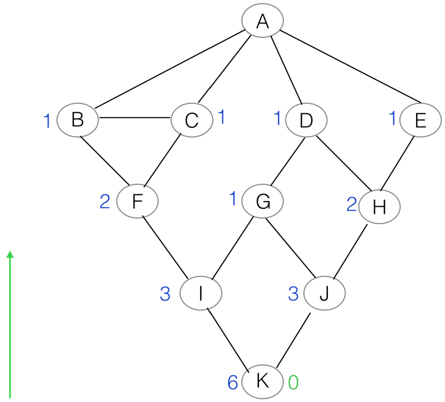
while S *not empty* **do**

  pop $w \leftarrow S$

  for $v \in Pred[w]$ **do** $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \cdot (1 + \delta[w])$

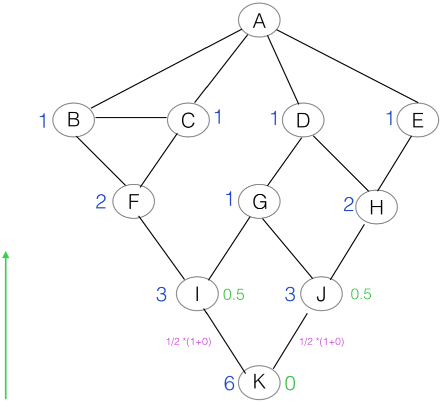
  if $w \neq s$ **then** $c_B[w] \leftarrow c_B[w] + \delta[w]$

Step 3 - Calculate $\delta(s|v)$, the dependency of s on v



$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

Step 3 - Calculate $\delta(s|v)$, the dependency of s on v



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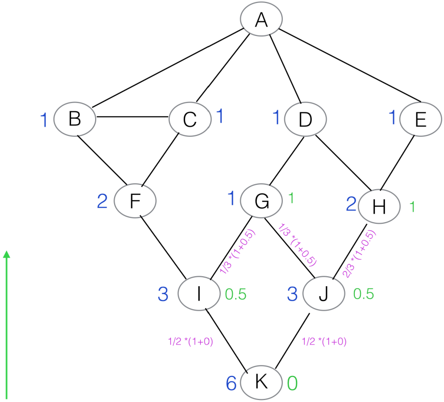

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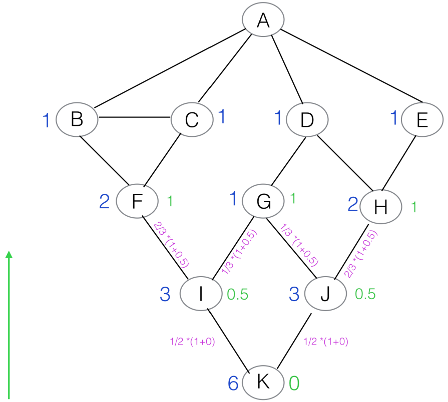

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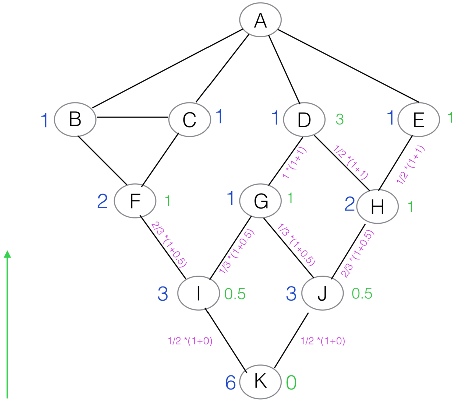

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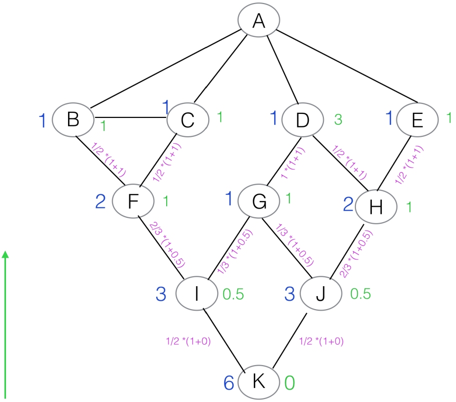

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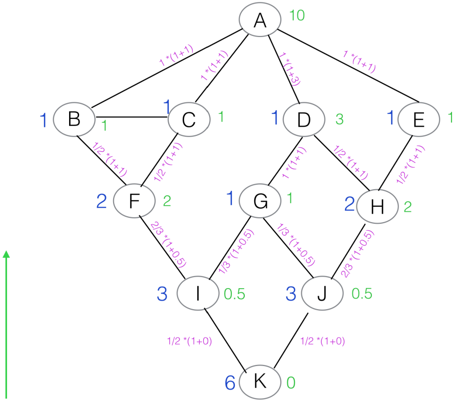

$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

Step 4 - Calculate betweenness centrality

- You saw one iteration with $s = A$.
- Now perform V iterations, once with each node as source.
- Sum up the $\delta(s|v)$ for each node: this gives the node's betweenness centrality.

Brandes (2008) pseudocode

Shortest-path vertex betweenness (Brandes, 2001).

```
input: directed graph G = (V, E)
data: queue Q, stack S (both initially empty)
      and for all v ∈ V:
      dist[v]: distance from source
      Pred[v]: list of predecessors on shortest paths from source
      σ[v]: number of shortest paths from source to v ∈ V
      δ[v]: dependency of source on v ∈ V
output: betweenness c_B[v] for all v ∈ V (initialized to 0)

for s ∈ V do
    ▼ single-source shortest-paths problem
        ▼ initialization
            for w ∈ V do Pred[w] ← empty list
            for t ∈ V do dist[t] ← ∞;  σ[t] ← 0
            dist[s] ← 0;  σ[s] ← 1
            enqueue s → Q

        while Q not empty do
            dequeue v ← Q;  push v → S
            foreach vertex w such that (v, w) ∈ E do
                ▼ path discovery  // — w found for the first time?
                    if dist[w] = ∞ then
                        dist[w] ← dist[v] + 1
                        enqueue w → Q
                ▼ path counting  // — edge (v, w) on a shortest path?
                    if dist[w] = dist[v] + 1 then
                        σ[w] ← σ[w] + σ[v]
                        append v → Pred[w]

    ▼ accumulation  // — back-propagation of dependencies
        for v ∈ V do δ[v] ← 0
        while S not empty do
            pop w ← S
            for v ∈ Pred[w] do δ[v] ← δ[v] + σ[v]/σ[w] · (1 + δ[w])
            if w ≠ s then c_B[w] ← c_B[w] + δ[w]
```

Brandes (2008): undirected graphs

- As specified, this is for directed graphs.
- But undirected graphs are easy: the algorithm works in exactly the same way, except that each pair is considered twice, once in each direction.
- Therefore: halve the scores at the end for undirected graphs.
- Brandes (2008) has lots of other variants, including edge betweenness centrality, which we'll use in the next session.

Today

- Task 11: Implement the Brandes algorithm for efficiently determining the betweenness of each node.

Literature

- Detailed notes on the Brandes algorithm on course page / Moodle.
- Easley and Kleinberg (2010, page 79-82). But this is an informal description.
- Ulrich Brandes (2001). A faster algorithm for betweenness centrality. *Journal of Mathematical Sociology*. 25:163–177.
- Ulrich Brandes (2008) On variants of shortest-path betweenness centrality and their generic computation. *Social Networks*. 30 (2008), pp. 136–145