

# Machine Learning and Bayesian Inference

## How to evaluate Gaussian integrals

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### 1 Introduction

The following notes show how to evaluate the standard integral required in deriving the approximation to the Bayes-optimal neural network.

### 2 Gaussian integrals: the simple case

The simplest version of the problem is to evaluate the integral

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx.$$

This is a fairly standard integration problem and several solutions are available in text books. For example, start by squaring it, so

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \exp\left(-\frac{ax^2}{2}\right) dx \times \int_{-\infty}^{\infty} \exp\left(-\frac{ay^2}{2}\right) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{a}{2}(x^2 + y^2)\right) dx dy. \end{aligned}$$

Then convert to polar co-ordinates, so  $x = r \cos \theta$ ,  $y = r \sin \theta$  and the Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

We now have

$$I^2 = \int_0^{2\pi} \int_0^{\infty} r \exp\left(-\frac{ar^2}{2}\right) dr d\theta$$

and as

$$-\frac{1}{a} \frac{d}{dr} \left( \exp\left(-\frac{ar^2}{2}\right) \right) = r \exp\left(-\frac{ar^2}{2}\right)$$

this is

$$I^2 = \int_0^{2\pi} \left[ -\frac{1}{a} \exp\left(-\frac{ar^2}{2}\right) \right]_0^{\infty} d\theta = \frac{1}{a} \int_0^{2\pi} d\theta = \frac{2\pi}{a}$$

and so

$$I = \sqrt{\frac{2\pi}{a}}.$$

### 3 Gaussian integrals: the general case

The problem now is to evaluate the more general integral

$$I = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} (\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + c) \right) d\mathbf{x}$$

where  $\mathbf{A}$  is an  $n \times n$  symmetric matrix with real-valued elements,  $\mathbf{b} \in \mathbb{R}^n$  is a real-valued vector and  $c \in \mathbb{R}$ . First of all, we can dispose of the constant part of the integrand as

$$I = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} (\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}) \right) \exp \left( -\frac{c}{2} \right) d\mathbf{x} = \exp \left( -\frac{c}{2} \right) I'$$

where

$$I' = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} (\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}) \right) d\mathbf{x}.$$

We're now going to make a change of variables, based on the fact that  $\mathbf{A}$  has  $n$  eigenvalues  $v_i$  and  $n$  eigenvectors  $\mathbf{e}_i$  such that

$$\mathbf{A} \mathbf{e}_i = v_i \mathbf{e}_i \tag{1}$$

for  $i = 1, \dots, n$ . The eigenvalues can be found such that they are orthonormal

$$\mathbf{e}_i^T \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying (1) on both sides by  $\mathbf{A}^{-1}$  gives

$$\mathbf{A}^{-1} \mathbf{A} \mathbf{e}_i = \mathbf{I}_n \mathbf{e}_i = \mathbf{e}_i = \mathbf{A}^{-1} v_i \mathbf{e}_i$$

for  $i = 1, \dots, n$ , where  $\mathbf{I}_n$  is the  $n \times n$  identity matrix. Consequently

$$\mathbf{A}^{-1} \mathbf{e}_i = \frac{1}{v_i} \mathbf{e}_i$$

for  $i = 1, \dots, n$  and  $\mathbf{A}^{-1}$  has the same eigenvectors as  $\mathbf{A}$ , but eigenvalues  $1/v_i$ . As the eigenvectors are orthonormal, any vector  $\mathbf{x}$  can be written as

$$\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{e}_i$$

for suitable values  $\lambda_i$ , and we can represent  $\mathbf{b}$  as

$$\mathbf{b} = \sum_{i=1}^n \beta_i \mathbf{e}_i$$

in the same way. Next, we make a change of variables from  $\mathbf{x}$  to

$$\boldsymbol{\lambda}^T = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}.$$

To make a change of variables we need to compute the Jacobian and rewrite the integral. The Jacobian for this transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial \lambda_1} & \frac{\partial x_2}{\partial \lambda_1} & \dots & \frac{\partial x_n}{\partial \lambda_1} \\ \frac{\partial x_1}{\partial \lambda_2} & \frac{\partial x_2}{\partial \lambda_2} & \dots & \frac{\partial x_n}{\partial \lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial \lambda_n} & \frac{\partial x_2}{\partial \lambda_n} & \dots & \frac{\partial x_n}{\partial \lambda_n} \end{vmatrix}.$$

As we saw above that

$$\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{e}_i$$

we have

$$x_j = \sum_{i=1}^n \lambda_i \mathbf{e}_i^{(j)}$$

where  $\mathbf{e}_i^{(j)}$  is the  $j$ th element of  $\mathbf{e}_i$ , and so

$$\frac{\partial x_j}{\partial \lambda_k} = \mathbf{e}_k^{(j)}.$$

Thus

$$J = \begin{vmatrix} \vdots & \vdots & \dots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ \vdots & \vdots & \dots & \vdots \end{vmatrix}.$$

That is, the determinant of the matrix having the eigenvectors as its columns. Define

$$\mathbf{E} = \begin{pmatrix} \vdots & \vdots & \dots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

such that  $J = |\mathbf{E}|$ . As the eigenvectors are orthonormal we have

$$J^2 = |\mathbf{E}||\mathbf{E}| = |\mathbf{E}||\mathbf{E}^T| = |\mathbf{E}\mathbf{E}^T| = |\mathbf{I}_n| = 1$$

and so  $J = 1$ .

Let's now look at the integrand

$$\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}.$$

Looking at the first term

$$\begin{aligned} \left( \sum_{i=1}^n \lambda_i \mathbf{e}_i^T \right) \mathbf{A} \left( \sum_{i=1}^n \lambda_i \mathbf{e}_i \right) &= \left( \sum_{i=1}^n \lambda_i \mathbf{e}_i^T \right) \left( \sum_{i=1}^n \lambda_i \mathbf{A} \mathbf{e}_i \right) \\ &= \left( \sum_{i=1}^n \lambda_i \mathbf{e}_i^T \right) \left( \sum_{i=1}^n \lambda_i v_i \mathbf{e}_i \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j v_i \mathbf{e}_j^T \mathbf{e}_i \\ &= \sum_{i=1}^n v_i \lambda_i^2. \end{aligned}$$

The second term simplifies in a similar way

$$\begin{aligned}\mathbf{b}^T \mathbf{x} &= \left( \sum_{i=1}^n \beta_i \mathbf{e}_i^T \right) \left( \sum_{j=1}^n \lambda_j \mathbf{e}_j \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \beta_i \lambda_j \mathbf{e}_i^T \mathbf{e}_j \\ &= \sum_{i=1}^n \beta_i \lambda_i\end{aligned}$$

and so the integrand becomes

$$\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} = \sum_{i=1}^n (v_i \lambda_i^2 + \beta \lambda_i).$$

Thus the result of changing the variable is that

$$\begin{aligned}I' &= \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} (\mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x}) \right) d\mathbf{x} \\ &= \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \left( \sum_{i=1}^n (v_i \lambda_i^2 + \beta \lambda_i) \right) \right) d\boldsymbol{\lambda} \\ &= \prod_{i=1}^n \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} (v_i \lambda_i^2 + \beta \lambda_i) \right) d\lambda_i.\end{aligned}$$

What have we gained by changing the variable?

- We have changed a multiple integral into a *product of single integrals*.
- Each of these single integrals is *almost* of a form that can be solved using the simple case above.

How do we proceed? Writing

$$\left( -\frac{1}{2} (v_i \lambda_i^2 + \beta \lambda_i) \right) = -\frac{v_i}{2} \left( \lambda_i + \frac{\beta}{2v_i} \right)^2 + \frac{\beta^2}{8v_i}$$

and changing the variable in the simple integral from  $\lambda_i$  to

$$\theta_i = \left( \lambda_i + \frac{\beta}{2v_i} \right)$$

gives

$$\frac{d\theta_i}{d\lambda_i} = 1$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} (v_i \lambda_i^2 + \beta \lambda_i) \right) d\lambda_i &= \exp \left( \frac{\beta^2}{8v_i} \right) \int_{-\infty}^{\infty} \exp \left( -\frac{v_i}{2} \theta_i^2 \right) d\theta_i \\ &= \exp \left( \frac{\beta^2}{8v_i} \right) \left( \frac{2\pi}{v_i} \right)^{1/2}\end{aligned}$$

using the simple case. We now have

$$I' = \prod_{i=1}^n \exp\left(\frac{\beta_i^2}{8v_i}\right) \left(\frac{2\pi}{v_i}\right)^{1/2}.$$

This can be simplified further in two steps. First, if  $\mathbf{A}$  has eigenvalues  $v_i$  then

$$|\mathbf{A}| = \prod_{i=1}^n v_i$$

and so

$$\prod_{i=1}^n \left(\frac{1}{v_i}\right) = |\mathbf{A}|^{-1}.$$

Thus

$$\prod_{i=1}^n \left(\frac{2\pi}{v_i}\right)^{1/2} = (2\pi)^{n/2} |\mathbf{A}|^{-1/2}.$$

Then, we have

$$\begin{aligned} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b} &= \left( \sum_{i=1}^n \beta_i \mathbf{e}_i^T \right) \mathbf{A}^{-1} \left( \sum_{i=1}^n \beta_i \mathbf{e}_i \right) \\ &= \left( \sum_{i=1}^n \beta_i \mathbf{e}_i^T \right) \left( \sum_{i=1}^n \frac{\beta_i}{v_i} \mathbf{e}_i \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \beta_j \mathbf{e}_j^T \mathbf{e}_i \frac{\beta_i}{v_i} \\ &= \sum_{i=1}^n \frac{\beta_i^2}{v_i}. \end{aligned}$$

Thus

$$\begin{aligned} \prod_{i=1}^n \exp\left(\frac{\beta_i^2}{8v_i}\right) &= \exp\left(\frac{1}{8} \sum_{i=1}^n \frac{\beta_i^2}{v_i}\right) \\ &= \exp\left(\frac{1}{8} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}\right) \end{aligned}$$

and collecting everything together we have,

$$\boxed{\begin{aligned} I &= \exp\left(-\frac{c}{2}\right) (2\pi)^{n/2} |\mathbf{A}|^{-1/2} \exp\left(\frac{1}{8} \mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}\right) \\ &= (2\pi)^{n/2} |\mathbf{A}|^{-1/2} \exp\left(-\frac{1}{2} \left(c - \frac{\mathbf{b}^T \mathbf{A}^{-1} \mathbf{b}}{4}\right)\right). \end{aligned}}$$