

MPhil Advanced Computer Science
Topics in Logic and Complexity

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Exercise Sheet 2

1. In the lecture we saw an illustration of a construction to show that *acyclicity* of graphs is not definable in first-order logic. Write out a proof of this result.

Prove that *acyclicity* is not definable in $\text{Mon}.\Sigma_1^1$. Is it definable in $\text{Mon}.\Pi_1^1$?

2. Prove (using Hanf's theorem or otherwise) that 3-colourability of graphs is not definable in first-order logic.

Graph 3-colourability (and, indeed, 2-colourability) are definable in $\text{Mon}.\Sigma_1^1$. Can you show they are not definable in $\text{Mon}.\Pi_1^1$? Are they definable in *universal second-order logic*?

3. Prove the lemma stated in the lecture that any formula that is positive in the relation symbol R defines a monotone operator.
4. Prove that the formula of LFP $\neg[\mathbf{lfp}_{R,\mathbf{x}}\neg\phi(R/\neg R)](\mathbf{x})$, where $\phi(R/\neg R)$ denotes the result of replacing all occurrences of R in ϕ by $\neg R$, defines the greatest fixed point of the operator defined by ϕ .

5. In the lectures, we saw how definitions by simultaneous induction can be replaced by a single application of the **lfp** operator. In this exercise, you are asked to show the same for *nested* applications of the **lfp** operator.

Suppose $\phi(\mathbf{x}, \mathbf{y}, S, T)$ is a formula in which the relational variables S (of arity s) and T (of arity t) only appear positively, and \mathbf{x} and \mathbf{y} are tuples of variables of length s and t respectively. Show that (for any t -tuple of terms \mathbf{t}) the predicate expression

$$[\mathbf{lfp}_{S,\mathbf{x}}([\mathbf{lfp}_{T,\mathbf{y}}\phi](\mathbf{t}))]$$

is equivalent to an expression with just one application of **lfp**.

6. Consider a vocabulary consisting of two unary relations P and O , one binary relation E and two constants s and t . We say that a structure $\mathbb{A} = (A, P, O, E, s, t)$ in this vocabulary is an *arena* if $P \cup O = A$ and $P \cap O = \emptyset$. That is, P and O partition the universe into two disjoint sets.

An arena defines the following game played between a *player* and an *opponent*. The game involves a *token* that is initially placed on the element s . At each move, if the token is currently on an element of P it is *player* who plays and if it is on an element of O , it is *opponent* who plays. At each move, if the token is on an element a , the one who plays chooses an element b such that $(a, b) \in E$ and moves the token from a to b . If the token reaches t at any point then *player* has won the game.

We define **GAME** to be the class of arenas for which *player* has a strategy for winning the game. Note that in an arena $\mathbb{A} = (A, P, O, E, s, t)$, *player* has a strategy to win from an element a if *either* $a \in P$ and there is some move from a so that *player* still has a strategy to win after that move *or* $a \in O$ and for every move from a , *player* can win after that move.

- (a) Give a sentence of **LFP** that defines the class of structures **GAME**.

We say that a collection \mathcal{C} of decision problems is *closed under logarithmic space reductions* if whenever $A \in \mathcal{C}$ and $B \leq_L A$ (i.e. B is reducible to A by a logarithmic-space reduction) then $B \in \mathcal{C}$.

The class of structures **GAME** defined above is known to be **P**-complete under logarithmic-space reductions.

- (b) Explain why this, together with (a) implies that the class of problems definable in **LFP** is *not* closed under logarithmic-space reductions.
7. Give a sentence of **LFP** that defines the class of linear orders with an even number of elements.
8. The *directed graph reachability problem* is the problem of deciding, given a structure (V, E, s, t) where E is an arbitrary binary relation on V , and $s, t \in V$, whether (s, t) is in the reflexive-transitive closure of E . This problem is known to be decidable in **NL**.

Transitive closure logic is the extension of first-order logic with an operator **tc** which allows us to form formulae

$$\phi \equiv [\mathbf{tc}_{\mathbf{x}, \mathbf{y}} \psi](\mathbf{t}_1, \mathbf{t}_2)$$

where \mathbf{x} and \mathbf{y} are k -tuples of variables and \mathbf{t}_1 and \mathbf{t}_2 are k -tuples of terms, for some k ; and all occurrences of variables \mathbf{x} and \mathbf{y} in ψ are bound in ϕ . The semantics is given by saying, if \mathbf{a} is an interpretation for the free variables of ϕ , then $\mathcal{A} \models \phi[\mathbf{a}]$ just in case $(\mathbf{t}_1^{\mathbf{a}}, \mathbf{t}_2^{\mathbf{a}})$ is in the reflexive-transitive closure of the binary relation defined by $\psi(\mathbf{x}, \mathbf{y})$ on A^k .

- (a) Show that any class of structures definable by a sentence ϕ , as above, where ψ is first-order, is decidable in **NL**.
- (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

is decidable in **NL**, then there is a sentence of transitive-closure logic that defines K .

9. For a binary relation E on a set A , define its *deterministic transitive closure* to be the set of pairs (a, b) for which there are $c_1, \dots, c_n \in A$ such

that $a = c_1$, $b = c_n$ and for each $i < n$, c_{i+1} is the *unique* element of A with $(c_i, c_{i+1}) \in E$.

Let DTC denote the logic formed by extending first-order logic with an operator **dtc** with syntax analogous to **tc** above, where $[\mathbf{dte}_{\mathbf{x},\mathbf{y}}\psi]$ defines the deterministic transitive closure of $\psi(\mathbf{x}, \mathbf{y})$.

- (a) Show that every sentence of DTC defines a class of structures decidable in L.
- (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and

$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$

is decidable in L, then there is a sentence of DTC that defines K .

- 10. Let \mathcal{E} be the class of graphs which are disjoint unions of cliques. That is, the vertices of each graph $G \in \mathcal{E}$ can be partitioned into cliques in such a way that there are no edges between the parts.

Show that satisfaction of FO is *fixed-parameter tractable* on \mathcal{E} .

- 11. Prove that every MSO sentence ϕ is equivalent, *over words* to a sentence of $\text{Mon.}\Sigma_1^1$.
- 12. In proving the Büchi-Elgot-Trakhtenbrot theorem, we used the fact that for every m , the relation \equiv_m^{MSO} is *right invariant*. Write a proof of this fact.