Lecture 15
Presheaf categories

Let $\mathbf{C}$ be a small category. The functor category $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$ is called the category of presheaves on $\mathbf{C}$.

- objects are contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.
Yoneda functor

\[ y : C \to \text{Set}^{\text{C}^{\text{op}}} \]

(where \( C \) is a small category)

is the Curried version of the \( \text{hom} \) functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set} \]
Yoneda functor

\[ y : C \to \text{Set}^{\text{op}} \]

(where \( C \) is a small category)

is the Curried version of the hom functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \overset{\text{Hom}_C}{\longrightarrow} \text{Set} \]

▶ For each \( C \)-object \( X \), the object \( yX \in \text{Set}^{\text{op}} \) is the functor \( C(\_, X) : C^{\text{op}} \to \text{Set} \) given by

\[
\begin{array}{cccc}
Z & \mapsto & C(Z, X) & \overset{g \circ f}{\mapsto} \\
\downarrow f & & \uparrow & \\
Y & \mapsto & C(Y, X) & \overset{g}{\mapsto}
\end{array}
\]
Yoneda functor

\[ y : C \to \text{Set}^{\text{C}^{\text{op}}} \]

(where \( C \) is a small category)

is the Curried version of the hom functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set} \]

▪️ For each \( C \)-object \( X \), the object \( yX \in \text{Set}^{\text{C}^{\text{op}}} \) is the functor \( C(\_ , X) : C^{\text{op}} \to \text{Set} \) given by

\[ \begin{align*}
Z & \mapsto C(Z, X) \\
Y & \mapsto C(Y, X)
\end{align*} \]

\[ f \mapsto g \circ f \]

\[ f^* \]

This function is often written as \( f^* \)
Yoneda functor

\[ y : \mathcal{C} \to \text{Set}^{\mathcal{C}^{\text{op}}} \]

(where \( \mathcal{C} \) is a small category)

is the Curried version of the \( \text{hom} \) functor

\[ \mathcal{C} \times \mathcal{C}^{\text{op}} \cong \mathcal{C}^{\text{op}} \times \mathcal{C} \overset{\text{Hom}_{\mathcal{C}}}{\longrightarrow} \text{Set} \]

For each \( \mathcal{C} \)-morphism \( Y \xrightarrow{f} X \), the morphism \( yY \xrightarrow{yf} yX \) in \( \text{Set}^{\mathcal{C}^{\text{op}}} \) is the natural transformation whose component at any given \( Z \in \mathcal{C}^{\text{op}} \) is the function

\[
\begin{align*}
yY(Z) & \xrightarrow{(yf)_Z} yX(Z) \\
\mathcal{C}(Z, Y) & \xrightarrow{=} \mathcal{C}(Z, X) \\
g & \xrightarrow{f \circ g}
\end{align*}
\]
Yoneda functor

\[ y : C \rightarrow \text{Set}^{\text{C}^{\text{op}}} \]

(where \( C \) is a small category)

is the Curried version of the hom functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set} \]

For each \( C \)-morphism \( Y \xrightarrow{f} X \), the morphism \( yY \xrightarrow{yf} yX \) in \( \text{Set}^{\text{C}^{\text{op}}} \) is the natural transformation whose component at any given \( Z \in C^{\text{op}} \) is the function

\[
yY(Z) \xrightarrow{(yf)_Z} yX(Z)
\]

\[
\text{C}(Z, Y) \xrightarrow{(yf)_Z} \text{C}(Z, X)
\]

this function is often written as \( f_* \)
The Yoneda Lemma

For each small category \( \mathbf{C} \), each object \( X \in \mathbf{C} \) and each presheaf \( F \in \text{Set}^{\mathbf{C}^{\text{op}}} \), there is a bijection of sets

\[
\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}} (yX, F) \cong F(X)
\]

which is natural in both \( X \) and \( F \).
The Yoneda Lemma

For each small category \( \mathcal{C} \), each object \( X \in \mathcal{C} \) and each presheaf \( F \in \text{Set}^{\mathcal{C}^{\text{op}}} \), there is a bijection of sets

\[
\eta_{X,F} : \text{Set}^{\mathcal{C}^{\text{op}}}((yX, F) \cong F(X)
\]

which is natural in both \( X \) and \( F \).

the set of natural transformations from the functor \( yX : \mathcal{C}^{\text{op}} \to \text{Set} \) to the functor \( F : \mathcal{C}^{\text{op}} \to \text{Set} \)

the value of \( F : \mathcal{C}^{\text{op}} \to \text{Set} \) at \( X \)
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \text{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \to F(X)$:

for each $\theta : yX \to F$ in $\text{Set}^{\mathbf{C}^{\text{op}}}$ we have the function

$\mathcal{C}(X, X) = yX(X) \xrightarrow{\theta_X} F(X)$ and define

$$\eta_{X,F}(\theta) \triangleq \theta_X(\text{id}_X)$$
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \text{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F}^{-1} : F(X) \rightarrow \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in yX(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\text{Set}$ and hence $F(f)(x) \in F(Y)$;
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \text{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F}^{-1} : F(X) \to \text{Set}^{\mathbf{C}^{\text{op}}}(yX, F)$:

for each $x \in F(X)$, $Y \in \mathbf{C}$ and $f \in yX(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\text{Set}$ and hence $F(f)(x) \in F(Y)$;

Define \( \left( \eta_{X,F}^{-1}(x) \right)_Y : yX(Y) \to F(Y) \) by

$$\left( \eta_{X,F}^{-1}(x) \right)_Y (f) \triangleq F(f)(x)$$

check this gives a natural transformation $\eta_{X,F}^{-1}(x) : yX \to F$
Proof of $\eta_{X,F} \circ \eta_{X,F}^{-1} = \text{id}_{F(X)}$

For any $x \in F(X)$ we have

$$\eta_{X,F} \left( \eta_{X,F}^{-1}(x) \right) \triangleq \left( \eta_{X,F}^{-1}(x) \right)_X \left( \text{id}_X \right)$$

by definition of $\eta_{X,F}$

$$\triangleq F(\text{id}_X)(x)$$

by definition of $\eta_{X,F}^{-1}$

$$= \text{id}_{F(X)}(x)$$

since $F$ is a functor

$$= x$$
Proof of  \[ \eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}_{\text{Set}^{\text{op}}(yX,F)} \]

For any \( yX \xrightarrow{\theta} F \) in \( \text{Set}^{\text{op}} \) and \( Y \xrightarrow{f} X \) in \( C \), we have

\[
\left( \eta_{X,F}^{-1} (\eta_{X,F}(\theta)) \right)_Y f \triangleq \left( \eta_{X,F}^{-1} (\theta_X(id_X)) \right)_Y f
\]

\[
\triangleq F(f)(\theta_X(id_X))
\]

\[
= \theta_Y(f^*(id_X))
\]

\[
\triangleq \theta_Y(id_X \circ f)
\]

\[
= \theta_Y(f)
\]

by definition of \( \eta_{X,F} \)

by definition of \( \eta_{X,F}^{-1} \)

by naturality of \( \theta \)

by definition of \( f^* \)

naturality of \( \theta \)
Proof of $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}_{\text{Set}^{\text{op}}(yX,F)}$

For any $yX \xrightarrow{\theta} F$ in $\text{Set}^{\text{op}}$ and $Y \xrightarrow{f} X$ in $C$, we have

$$(\eta_{X,F}^{-1}(\eta_{X,F}(\theta)))_Y f \triangleq (\eta_{X,F}^{-1}(\theta_X(id_X)))_Y f$$

$\triangleq F(f)(\theta_X(id_X))$ \hspace{1cm} by definition of $\eta_{X,F}$

$= \theta_Y(f^*(id_X))$ \hspace{1cm} by naturality of $\theta$

$\triangleq \theta_Y(id_X \circ f)$ \hspace{1cm} by definition of $f^*$

$= \theta_Y(f)$

so $\forall \theta, Y, (\eta_{X,F}^{-1}(\eta_{X,F}(\theta)))_Y = \theta_Y$

so $\forall \theta, \eta_{X,F}^{-1}(\eta_{X,F}(\theta)) = \theta$

so $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}$. 
The Yoneda Lemma

For each small category $\mathcal{C}$, each object $X \in \mathcal{C}$ and each presheaf $F \in \text{Set}^{\mathcal{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathcal{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$. 
Proof that $\eta_{X,F}$ is natural in $F$:

Given $F \xrightarrow{\varphi} G$ in $\text{Set}^{\text{op}}$, have to show that

$$\begin{align*}
\text{Set}^{\text{op}}(yX, F) & \xrightarrow{\eta_{X,F}} F(X) \\
\text{Set}^{\text{op}}(yX, G) & \xrightarrow{\eta_{X,G}} G(X)
\end{align*}$$

commutes in $\text{Set}$. For all $yX \xrightarrow{\theta} F$ we have

$$\varphi_X(\eta_{X,F}(\theta)) \triangleq \varphi_X(\theta_X(\text{id}_X))$$

$$\triangleq (\varphi \circ \theta)_X(\text{id}_X)$$

$$\triangleq \eta_{X,G}(\varphi \circ \theta)$$

$$\triangleq \eta_{X,G}(\varphi_*(\theta))$$
Proof that $\eta_{X,F}$ is natural in $X$:

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, have to show that

\[
\begin{align*}
\text{Set}^{\mathbf{C}^{\text{op}}}(yX, F) & \xrightarrow{\eta_{X,F}} F(X) \\
(yf)^* & \downarrow \quad \downarrow F(f) \\
\text{Set}^{\mathbf{C}^{\text{op}}}(yY, F) & \xrightarrow{\eta_{Y,F}} F(Y)
\end{align*}
\]

commutes in $\text{Set}$. For all $yX \xrightarrow{\theta} F$ we have

\[
F(f)((\eta_{X,F}(\theta))) \triangleq F(f)(\theta_X(id_X))
\]
\[
= \theta_Y(f^*(id_X))
\]
\[
= \theta_Y(f)
\]
\[
= \theta_Y(f)(id_Y)
\]
\[
\triangleq (\theta \circ yf)(id_Y)
\]
\[
\triangleq \eta_{Y,F}(\theta \circ yf)
\]
\[
\triangleq \eta_{Y,F}((yf)^*(\theta))
\]

by naturality of $\theta$
Corollary of the Yoneda Lemma:

the functor \( y : C \rightarrow \text{Set}^{\text{C}^{\text{op}}} \) is full and faithful.

In general, a functor \( F : C \rightarrow D \) is

- **faithful** if for all \( X, Y \in C \) the function
  \[
  \text{C}(X, Y) \rightarrow \text{D}(F(X), F(Y))
  \]
  \[
  f \mapsto F(f)
  \]
  is injective:
  \[
  \forall f, f' \in \text{C}(X, Y), \ F(f) = F(f') \Rightarrow f = f'
  \]

- **full** if the above functions are all surjective:
  \[
  \forall g \in \text{D}(F(X), F(Y)), \exists f \in \text{C}(X, Y), \ F(f) = g
  \]