Lecture 14

Dependent Types

A brief look at some category theory for modelling type theories with dependent types.

Will restrict attention to the case of **Set**, rather than in full generality.

Further reading:
M. Hofmann, *Syntax and Semantics of Dependent Types*. In: A.M. Pitts and P. Dybjer (eds), *Semantics and Logics of Computation* (CUP, 1997).

Simple types

$$\diamond, x_1 : T_1, \ldots, x_n : T_n \vdash t(x_1, \ldots, x_n) : T$$

Dependent types

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and more generally

$$\diamond,\ x_1 : T_1,\ x_2 : T_2(x_1),\ x_3 : T_3(x_1, x_2),\ \ldots \vdash t(x_1, x_2, x_3, \ldots) : T(x_1, x_2, x_3, \ldots)$$

If type expressions denote sets, then

$$\text{a type } T(x) \text{ dependent upon } x : I$$

should denote

$$\text{an indexed family of sets } (E\, i \mid i \in I)$$

i.e. $E : I \to \mathbf{Set}$ is a set-valued function on a set I.

For each $I \in \mathbf{Set}$, let $\boxed{\mathbf{Set}^I}$ be the category with

- $\mathtt{obj}(\mathbf{Set}^I) \triangleq (\mathtt{obj}\,\mathbf{Set})^I$, so objects are I-indexed families of sets, $X = (X_i \mid i \in I)$
- morphisms $f : X \to Y$ in $\mathbf{Set}^I$ are I-indexed families of functions $f = (f_i \in \mathbf{Set}(X_i, Y_i) \mid i \in I)$
- composition: $(g \circ f) \triangleq (g_i \circ f_i \mid i \in I)$

  (i.e. use composition of functions in $\mathbf{Set}$ at each index $i \in I$)
- identity: $\mathtt{id}_X \triangleq (\mathtt{id}_{X_i} \mid i \in I)$

  (i.e. use identity functions in $\mathbf{Set}$ at each index $i \in I$)

For each $p : I \to J$ in **Set**, let $\boxed{p^* : \mathbf{Set}^J \to \mathbf{Set}^I}$ be the functor defined by:

$$p^* \left(\begin{array}{c|c} Y_j & \\ \downarrow f_j & j \in J \\ Y'_j & \end{array} \right) \triangleq \left(\begin{array}{c|c} Y_{p\,i} & \\ \downarrow f_{p\,i} & i \in I \\ Y'_{p\,i} & \end{array} \right)$$

i.e. p^* takes J-indexed families of sets/functions to I-indexed ones by precomposing with p

Dependent products

of families of sets

For $I, j \in \mathbf{Set}$, consider the functor $\pi_1^* : \mathbf{Set}^I \to \mathbf{Set}^{I \times J}$ induced by precomposition with the first projection function $\pi_1 : I \times J \to I$.

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \to \mathbf{Set}^I$.

Proof. We apply the Theorem from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Sigma\, E \in \mathbf{Set}^I$ and $\eta_E : E \to \pi_1^*(\Sigma\, E)$ in $\mathbf{Set}^{I \times J}$ with the required universal property...

Recall :

$G : \mathbf{C} \leftarrow \mathbf{D}$ has a left adjoint iff for all $X \in \mathbf{C}$ there are $F\,X \in \mathbf{D}$ and $\eta_X \in \mathbf{C}(X, G(F\,X))$ with the universal property:

for all $Y \in \mathbf{D}$ and $f \in \mathbf{C}(X, G\,Y)$
there is a unique $\overline{f} \in \mathbf{D}(F\,X, Y)$
satisfying $G\,\overline{f} \circ \eta_X = f$

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \to \mathbf{Set}^I$.

For each $E \in \mathbf{Set}^{I \times J}$, define $\Sigma\, E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

$$(\Sigma\, E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{(j, e) \mid j \in J \wedge e \in E_{(i,j)}\}$$

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and define $\eta_E : E \to \pi_1^*(\Sigma\, E)$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i, j) \in I \times J$ to the function $(\eta_E)_{(i.j)} : E_{(i,j)} \to (\Sigma\, E)_i$ given by $\boxed{e \mapsto (j, e)}$.

Universal property–

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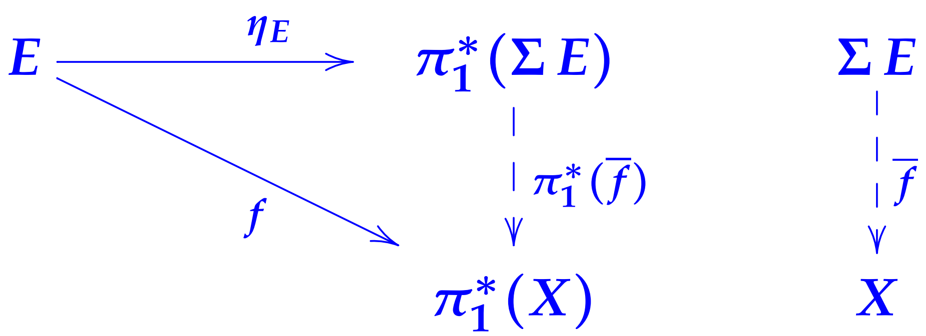
Universal property–*existence part*: given any $X \in \mathbf{Set}^I$ and $f : E \to \pi_1^*(X)$ in $\mathbf{Set}^{I\times J}$, we have



where for all $i \in I$, $j \in J$ and $e \in E_{(i,j)}$ $\overline{f}_i(j,e) \triangleq f_{(i,j)}(e)$

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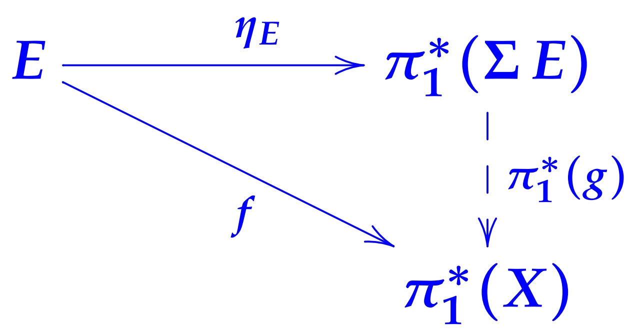
Universal property–*uniqueness part*: given $g : \Sigma\, E \to X$ in $\mathbf{Set}^I$ making



commute in $\mathbf{Set}^{I\times J}$,

then for all $i \in I$, and $(j,e) \in (\Sigma\, E)_i$ we have

$$\overline{f}_i(j,e) \triangleq f_{(i,j)}(e) = (\pi_1^* g \circ \eta_E)_{(i,j)}\, e = (\pi_1^* g)_{(i,j)}((\eta_E)_{(i,j)}\, e) \triangleq g_i(j,e)$$

so $g = \overline{f}$. $\square$

Dependent functions

of families of sets

We have seen that the left adjoint to $\pi_1^* : \mathbf{Set}^I \to \mathbf{Set}^{I \times J}$ is given by dependent products of sets.

Dually, dependent function sets give:

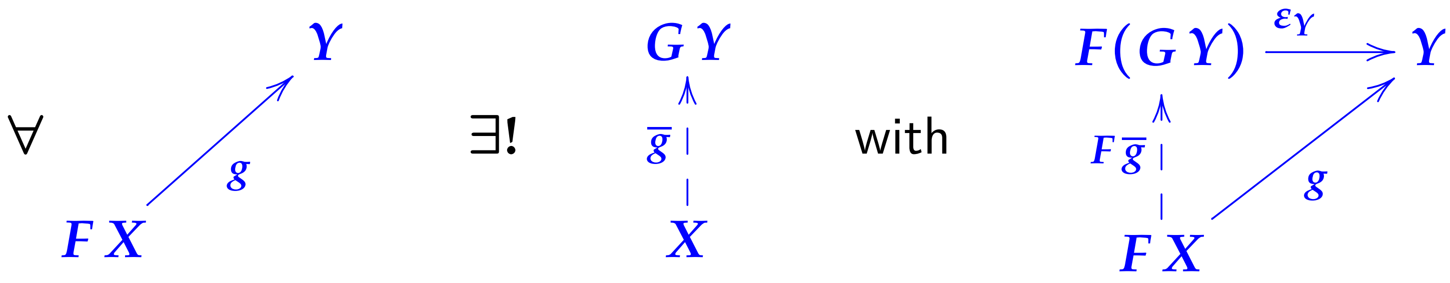
Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \longrightarrow \mathbf{Set}^I$.

Proof. We apply the Theorem from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Pi\, E \in \mathbf{Set}^I$ and $\varepsilon_E : \pi_1^*(\Pi\, E) \to E$ in $\mathbf{Set}^{I \times J}$ with the required universal property. . .

Recall:

Theorem. A functor $F : \mathbf{C} \to \mathbf{D}$ has a right adjoint iff for all $\mathbf{D}$-objects $Y \in \mathbf{D}$, there is a $\mathbf{C}$-object $G\,Y \in \mathbf{C}$ and a $\mathbf{C}$-morphism $\varepsilon_Y : F(G\,Y) \to Y$ with the following "universal property":

(UP) for all $X \in \mathbf{C}$ and $g \in \mathbf{D}(F\,X, Y)$ there is a unique $\overline{g} \in \mathbf{C}(X, G\,Y)$ satisfying $\varepsilon_Y \circ F(\overline{g}) = g$



Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \to \mathbf{Set}^{I}$.

For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi\, E \in \mathbf{Set}^{I}$ to be the function mapping each $i \in I$ to the set

$$(\Pi\, E)_i \triangleq \prod_{j \in J} E_{(i,j)} = \{f \subseteq (\Sigma\, E)_i \mid f \text{ is single-value and total}\}$$

where $f \subseteq (\Sigma\, E)_i$ is

single-valued if $\forall j \in J, \forall e, e' \in E_{(i,j)},\ (j, e) \in f \wedge (j, e') \in f \Rightarrow e = e'$

total if $\forall j \in J, \exists e \in E_{(i,j)}\ (j, e) \in f$

Thus each $f \in (\Pi\, E)_i$ is a dependently typed function mapping elements $j \in J$ to elements of $E_{(i,j)}$ (result set depends on the argument j).

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and define $\varepsilon_E : \pi_1^*(\Pi\, E) \to E$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i, j) \in I \times J$ to the function $(\varepsilon_E)_{(i,j)} : (\Pi\, E)_i \to E_{(i,j)}$ given by $\boxed{f \mapsto f\, j}$ = unique $e \in E_{(i,j)}$ such that $(j, e) \in f$.

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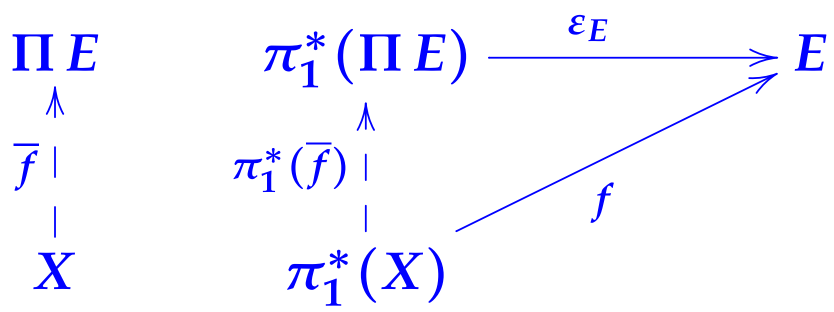

where for all $i \in I$ and $x \in X_i$ $\boxed{\overline{f}_i\, x \triangleq \{(j, f_{(i,j)}\, x) \mid j \in J\}}$

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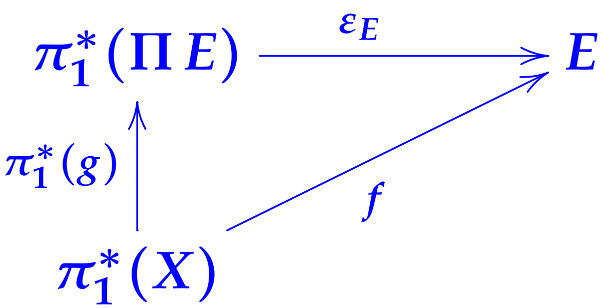

commute in $\mathbf{Set}^{I\times J}$,

then for all $i \in I$, $j \in J$ and $x \in X_i$ we have

$$\overline{f}_i\, x\, j \triangleq f_{(i,j)}\, x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)}\, x = (\varepsilon_E)_{(i,j)}(g_i\, x) \triangleq g_i\, x\, j$$

so $g = \overline{f}$. $\square$

Equivalence

Two categories $\mathbb{C}$ & $\mathbb{D}$ are isomorphic if they are isomorphic objects in the category of categories (of some size), that is, there are

functors

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$$

satisfying

$$Id_{\mathbb{C}} = G \circ F$$
$$F \circ G = Id_{\mathbb{D}}$$

(in which case, as usual, we write $\mathbb{C} \cong \mathbb{D}$)

Equivalence

Two categories $\mathbb{C}$ & $\mathbb{D}$ are equivalent if there are

functors & natural isomorphisms

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D} \qquad \begin{aligned} \eta &: \mathrm{Id}_{\mathbb{C}} \cong G \circ F \\ \varepsilon &: F \circ G \cong \mathrm{Id}_{\mathbb{D}} \end{aligned}$$

in which case one writes $\boxed{\mathbb{C} \simeq \mathbb{D}}$

Some deep results in mathematics take the form of equivalences:

E.g.

Stone duality: $\begin{pmatrix}\text{Category} \\ \text{of Boolean} \\ \text{algebras}\end{pmatrix}^{op} \simeq$ Category of compact, tot. disconn. Hausdorff spaces

Gelfand duality: $\begin{pmatrix}\text{abelian} \\ C^{*}\text{algebras}\end{pmatrix}^{op} \simeq$ Compact Hausdorff Spaces

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

$\mathbf{Set}/I$ is a slice category [Ex. Sh. 4, qu. 6]

- objects are (X, f) where $f \in \mathbf{Set}(X, I)$
- morphisms $g : (X, f) \to (X', f')$ are $g \in \mathbf{Set}(X, X')$ satisfying $f' \circ g = f$ in $\mathbf{Set}$
- composition & identities – as for $\mathbf{Set}$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

functor $F : \mathbf{Set}^I \longrightarrow \mathbf{Set}/I$

on objects: $$F(X) \triangleq \left(\begin{array}{c} \{(i,x) \mid i \in I \,\&\, x \in X_i\} \\ \big\downarrow \scriptstyle fst \\ I \end{array} \right)$$

on morphisms:

$$F(X \xrightarrow{f} X') \triangleq \begin{array}{ccc} \{(i,x) \mid i \in I \,\&\, x \in X_i\} & \longrightarrow & \{(i,x') \mid i \in I \,\&\, x' \in X'_i\} \\ & \searrow \quad I \quad \swarrow & \\ (i,x) & \longmapsto & (i, f_i\, x) \end{array}$$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

functor $G : \mathbf{Set}/I \to \mathbf{Set}^I$

on objects:

$$G\left(\begin{array}{c} E \\ \downarrow p \\ I \end{array}\right) \triangleq \left(\{e \in E \mid pe = i\} \mid i \in I\right)$$

on morphisms:

$$G\left(\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & I & \end{array}\right) \triangleq GE \xrightarrow{Gf} GE'$$

where for each $i \in I$

$$(Gf)_i\, e \triangleq f(e)$$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

There are natural isomorphisms

$$\eta : \mathrm{Id}_{\mathbf{Set}^I} \cong G \circ F$$

$$\varepsilon : F \circ G \cong \mathrm{Id}_{\mathbf{Set}/I}$$

defined by ... [exercise]

FACT Given $p : I \to J$ in $\mathbf{Set}$,

$$\mathbf{Set}/J \simeq \mathbf{Set}^J \xrightarrow{p^*} \mathbf{Set}^I \simeq \mathbf{Set}/I$$

is the functor "pullback along p"

Can generalize from $\mathbf{Set}$ to any category $\mathbb{C}$ with pullbacks & model Σ/Π types by left/right adjoints to pullback functor – see locally cartesian closed categories in literature.