Lecture 13
Recall:

Given categories and functors $\mathbf{C} \xleftarrow{F} \xrightarrow{G} \mathbf{D}$, an adjunction $\mathbf{F} \dashv \mathbf{G}$ is specified by functions

\[
\begin{align*}
\theta_{X,Y}: & \quad FX \xrightarrow{g} Y \\
& \quad X \xrightarrow{\bar{g}} GY
\end{align*}
\]

\[
\begin{align*}
\theta_{X,Y}^{-1}: & \quad FX \xrightarrow{f} Y \\
& \quad X \xrightarrow{f} GY
\end{align*}
\]

(for each $X \in \mathbf{C}$ and $Y \in \mathbf{D}$) satisfying $\bar{f} = f$, $\bar{g} = g$ and

\[
\begin{align*}
\begin{array}{c}
FX' \xrightarrow{Fu} FX \\
\xrightarrow{g} Y \\
x' \xrightarrow{u} X \xrightarrow{\bar{g}} GY
\end{array}
& \\
\begin{array}{c}
FX \xrightarrow{g} Y \xrightarrow{v} Y' \\
\xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'
\end{array}
\end{align*}
\]
**Theorem.** A category $\mathbf{C}$ has binary products iff the diagonal functor $\Delta = \langle \text{id}_\mathbf{C}, \text{id}_\mathbf{C} \rangle : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ has a right adjoint.

**Theorem.** A category $\mathbf{C}$ with binary products also has all exponentials of pairs of objects iff for all $X \in \mathbf{C}$, the functor $(_-) \times X : \mathbf{C} \to \mathbf{C}$ has a right adjoint.

Both these theorems are instances of the following theorem, a very useful characterisation of when a functor has a right adjoint (or dually, a left adjoint).
**Theorem.** A functor $F : C \rightarrow D$ has a right adjoint iff for all $D$-objects $Y \in D$, there is a $C$-object $G Y \in C$ and a $C$-morphism $\varepsilon_Y : F(G Y) \rightarrow Y$ with the following “universal property”:

\[
\text{(UP) for all } X \in C \text{ and } g \in D(FX, Y) \text{ there is a unique } \overline{g} \in C(X, G Y) \text{ satisfying } \varepsilon_Y \circ F(\overline{g}) = g
\]
Theorem. A functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint iff for all $\mathcal{D}$-objects $Y \in \mathcal{D}$, there is a $\mathcal{C}$-object $G Y \in \mathcal{C}$ and a $\mathcal{C}$-morphism $\varepsilon_Y : F(G Y) \to Y$ with the following “universal property”:

(UP) \hspace{1cm} \text{for all } X \in \mathcal{C} \text{ and } g \in \mathcal{D}(F X, Y) \text{ there is a unique } \overline{g} \in \mathcal{C}(X, G Y) \text{ satisfying } \varepsilon_Y \circ F(\overline{g}) = g
Theorem. A functor $F : C \to D$ has a right adjoint iff for all $D$-objects $Y \in D$, there is a $C$-object $GY \in C$ and a $C$-morphism $\varepsilon_Y : F(GY) \to Y$ with the following "universal property":

\[
\text{(UP)} \quad \begin{aligned}
\text{for all } X \in C \text{ and } g \in D(FX, Y) \\
\text{there is a unique } \bar{g} \in C(X, GY)
\end{aligned}
\]

satisfying $\varepsilon_Y \circ F(\bar{g}) = g$.
Theorem. A functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint iff for all $\mathcal{D}$-objects $Y \in \mathcal{D}$, there is a $\mathcal{C}$-object $GY \in \mathcal{C}$ and a $\mathcal{C}$-morphism $\epsilon_Y : F(GY) \to Y$ with the following “universal property”:

\begin{align*}
\text{(UP)} \\
\text{for all } X \in \mathcal{C} \text{ and } g \in \mathcal{D}(FX,Y) \\
\text{there is a unique } \overline{g} \in \mathcal{C}(X,GY) \\
\text{satisfying } \epsilon_Y \circ F(\overline{g}) = g
\end{align*}
Proof of the Theorem—“only if” part:

Given an adjunction $(F, G, \theta)$, for each $Y \in D$ we produce $\varepsilon_Y : F(GY) \to Y$ in $D$ satisfying (UP).
Proof of the **Theorem**—“only if” part:

Given an adjunction \((F, G, \theta)\), for each \(Y \in D\) we produce \(\varepsilon_Y : F(GY) \to Y\) in \(D\) satisfying \((UP)\).

We are given \(\theta_{X,Y} : D(FX, Y) \cong C(X, GY)\), natural in \(X\) and \(Y\). Define

\[
\varepsilon_Y \triangleq \theta^{-1}_{GY,Y}(\text{id}_{GY}) : F(GY) \to Y
\]

In other words \(\varepsilon_Y = \text{id}_{GY}\).
Proof of the Theorem—“only if” part:

Given an adjunction \((F, G, \theta)\), for each \(Y \in D\) we produce \(\varepsilon_Y : F(GY) \to Y\) in \(D\) satisfying (UP).

We are given \(\theta_{X,Y} : D(FX, Y) \cong C(X, GY)\), natural in \(X\) and \(Y\). Define

\[
\varepsilon_Y \triangleq \theta_{G Y, Y}^{-1}(\text{id}_{GY}) : F(GY) \to Y
\]

In other words \(\varepsilon_Y = \text{id}_{GY}\).

Given any \(\begin{cases} g : FX \to Y & \text{in } D \\ f : X \to GY & \text{in } C \end{cases}\), by naturality of \(\theta\) we have

\[
\begin{align*}
FX & \xrightarrow{g} Y \\
X & \xrightarrow{\overline{g}} GY
\end{align*}
\]

and

\[
\begin{align*}
\varepsilon_Y \circ Ff : FX & \xrightarrow{Ff} F(GY) \xrightarrow{\text{id}_{GY}} Y \\
f : X & \xrightarrow{f} GY \xrightarrow{\text{id}_{GY}} GY
\end{align*}
\]

Hence \(g = \varepsilon_Y \circ F\overline{g}\) and \(g = \varepsilon_Y \circ Ff \Rightarrow \overline{g} = f\).

Thus we do indeed have (UP).
Proof of the Theorem—“if” part:

We are given $F : C \to D$ and for each $Y \in D$ a $C$-object $GY$ and $C$-morphism $\varepsilon_Y : F(GY) \to Y$ satisfying (UP). We have to

1. extend $Y \mapsto GY$ to a functor $G : D \to C$

2. construct a natural isomorphism

$\theta : \text{Hom}_D \circ (F^{\text{op}} \times \text{id}_D) \cong \text{Hom}_C \circ (\text{id}_{C^{\text{op}}} \times G)$
Proof of the Theorem—“if” part:

We are given $F : C \to D$ and for each $Y \in D$ a $C$-object $GY$ and $C$-morphism $\varepsilon_Y : F(GY) \to Y$ satisfying $(UP)$. We have to

1. extend $Y \mapsto GY$ to a functor $G : D \to C$

For each $D$-morphism $g : Y' \to Y$ we get $F(GY') \xrightarrow{\varepsilon_Y'} Y' \xrightarrow{g} Y$ and can apply $(UP)$ to get

$$Gg \triangleq g \circ \varepsilon_Y' : GY' \to GY$$

The uniqueness part of $(UP)$ implies

$$G \text{id} = \text{id} \quad \text{and} \quad G(g' \circ g) = Gg' \circ Gg$$

so that we get a functor $G : D \to C$. □
Proof of the Theorem—“if” part:

We are given $F : \mathbf{C} \to \mathbf{D}$ and for each $Y \in \mathbf{D}$ a $\mathbf{C}$-object $GY$ and $\mathbf{C}$-morphism $\varepsilon_Y : F(GY) \to Y$ satisfying (UP). We have to

2. *construct a natural isomorphism*

$$\theta : \text{Hom}_\mathbf{D} \circ (F^{\text{op}} \times \text{id}_\mathbf{D}) \cong \text{Hom}_\mathbf{C} \circ (\text{id}_{\mathbf{C}^{\text{op}}} \times G)$$

Since for all $g : FX \to Y$ there is a unique $f : X \to GY$ with $g = \varepsilon_Y \circ Ff$,

$$f \mapsto \bar{f} \triangleq \varepsilon_Y \circ Ff$$

determines a bijection $\mathbf{C}(X, GY) \cong \mathbf{C}(FX, Y)$; and it is natural in $X$ & $Y$ because

$$Gv \circ f \circ u \triangleq \varepsilon_Y' \circ F(Gv \circ f \circ u)$$

$$= (\varepsilon_Y' \circ F(Gv)) \circ Ff \circ Fu \quad \text{since } F \text{ is a functor}$$

$$= (v \circ \varepsilon_Y) \circ Ff \circ Fu \quad \text{by definition of } Gv$$

$$= v \circ \bar{f} \circ Fu \quad \text{by definition of } \bar{f}$$

So we can take $\theta$ to be the inverse of this natural isomorphism. \[\square\]
Dual of the Theorem:

$G : \mathcal{C} \leftarrow \mathcal{D}$ has a left adjoint iff for all $X \in \mathcal{C}$ there are $FX \in \mathcal{D}$ and $\eta_X \in \mathcal{C}(X, G(FX))$ with the universal property:

\begin{align*}
\text{for all } Y \in \mathcal{D} \text{ and } f \in \mathcal{C}(X, GY) \\
\text{there is a unique } \overline{f} \in \mathcal{D}(FX, Y) \\
\text{satisfying } G \overline{f} \circ \eta_X = f
\end{align*}
**Dual of the Theorem:**

$G : C \leftarrow D$ has a left adjoint iff for all $X \in C$ there are $FX \in D$ and $\eta_X \in C(X, G(FX))$ with the universal property:

\[
\text{for all } Y \in D \text{ and } f \in C(X, GY) \text{ there is a unique } \bar{f} \in D(FX, Y) \text{ satisfying } G\bar{f} \circ \eta_X = f.
\]

E.g. we can conclude that the forgetful functor $U : \text{Mon} \rightarrow \text{Set}$ has a left adjoint $F : \text{Set} \rightarrow \text{Mon}$, because of the universal property of

\[
F\Sigma \triangleq (\text{List } \Sigma, @, \text{nil}) \quad \text{and} \quad \eta_\Sigma : \Sigma \rightarrow \text{List } \Sigma
\]

noted in Lecture 3.
Why are adjoint functors important/useful?

Their universal property (UP) usually embodies some useful mathematical construction

(e.g. “freely generated structures are left adjoints for forgetting-structure”)

and pins it down uniquely up to isomorphism.