TIMETABLE

Lecture #14 Tues 26 Nov

No lecture Thurs 28 Nov

Lecture #15 Tues 3 Dec

Lecture #16 Thurs 5 Dec 10am
Exercise Sheet 4 (graded)
anonymous results

highest = 40/40 (100%)
lowest = 28/40 (70%)
median = 35/40 (88%)
sample size = 15
Lecture 13
Recall:

Given categories and functors $\textbf{C} \xrightarrow{F} \xleftarrow{G} \textbf{D}$, an adjunction $\textbf{F} \dashv \textbf{G}$ is specified by functions

$$\xymatrix{ FX \ar[r]^g & Y \ar[d]_{\theta_{X,Y}} \ar@/^1pc/[r]^F & X \ar[r]^g & G Y}$$

$$\xymatrix{ FX \ar[r]^f & Y \ar[u]^{\theta_{X,Y}^{-1}} \ar@/^1pc/[r]^F & X \ar[r]^f & G Y}$$

(for each $X \in \textbf{C}$ and $Y \in \textbf{D}$) satisfying $\overline{f} = f$, $\overline{g} = g$ and

$$\xymatrix{ FX' \ar[r]^{Fu} & FX \ar[r]^g & Y \ar[d]_{\overline{g}} \ar@/^1pc/[r]^{F} & X' \ar[r]^{u} \ar[r]^\overline{g} & G Y}$$

$$\xymatrix{ FX \ar[r]^g & Y \ar[r]^v & Y' \ar[d]_{\overline{g}} \ar@/^1pc/[r]^{G} & X \ar[r]^\overline{g} \ar[r]^{Gv} & G Y' \ar[r] & G Y'}$$
Characterisation of right adjoints

**Theorem.** A functor $F : C \to D$ has a right adjoint iff for all $D$-objects $Y \in D$, there is a $C$-object $GY \in C$ and a $D$-morphism $\varepsilon_Y : F(GY) \to Y$ with the following “universal property”:

\[ \text{(UP)} \]

for all $X \in C$ and $g \in D(FX,Y)$ there is a unique $\overline{g} \in C(X,GY)$ satisfying $\varepsilon_Y \circ F(\overline{g}) = g$.
**Theorem.** A functor $F : C \to D$ has a right adjoint iff for all $D$-objects $Y \in D$, there is a $C$-object $GY \in C$ and a $D$-morphism $\varepsilon_Y : F(GY) \to Y$ with the following "universal property":

\[ \forall X \in C \text{ and } g \in D(FX, Y) \text{ there is a unique } \overline{g} \in C(X, GY) \text{ satisfying } \varepsilon_Y \circ F(\overline{g}) = g \]
Characterisation of right adjoints

**Theorem.** A functor $F : C \to D$ has a right adjoint iff for all $D$-objects $Y \in D$, there is a $C$-object $GY \in C$ and a $D$-morphism $\varepsilon_Y : F(GY) \to Y$ with the following “universal property”:

\[(UP)\]

for all $X \in C$ and $g \in D(FX, Y)$ there is a unique $\bar{g} \in C(X, GY)$ satisfying $\varepsilon_Y \circ F(\bar{g}) = g$
Theorem. A functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint iff for all $\mathcal{D}$-objects $Y \in \mathcal{D}$, there is a $\mathcal{C}$-object $GY \in \mathcal{C}$ and a $\mathcal{D}$-morphism $\varepsilon_Y : F(GY) \to Y$ with the following "universal property":

\[(\text{UP})\] for all $X \in \mathcal{C}$ and $g \in \mathcal{D}(FX, Y)$ there is a unique $\bar{g} \in \mathcal{C}(X, GY)$ satisfying $\varepsilon_Y \circ F(\bar{g}) = g$
Proof of the Theorem—“only if” part:

Given an adjunction \((F, G, \theta)\), for each \(Y \in D\) we produce \(\varepsilon_Y : F(GY) \to Y\) in \(D\) satisfying (UP).
Proof of the **Theorem**—“only if” part:

Given an adjunction \((F, G, \theta)\), for each \(Y \in \mathcal{D}\) we produce \(\varepsilon_Y : F(GY) \to Y\) in \(\mathcal{D}\) satisfying \((\text{UP})\).

We are given \(\theta_{X,Y} : \mathcal{D}(FX,Y) \cong \mathcal{C}(X,GY)\), natural in \(X\) and \(Y\). Define

\[
\varepsilon_Y \triangleq \theta_{G,Y,Y}^{-1}(\text{id}_{GY}) : F(GY) \to Y
\]

In other words \(\varepsilon_Y = \text{id}_{GY}\).
Proof of the Theorem—“only if” part:

Given an adjunction $(F, G, \theta)$, for each $Y \in D$ we produce $\varepsilon_Y : F(GY) \to Y$ in $D$ satisfying (UP).

We are given $\theta_{X,Y} : D(FX,Y) \cong C(X,GY)$, natural in $X$ and $Y$. Define

$$\varepsilon_Y \triangleq \theta_{GY,Y}^{-1}(\text{id}_{GY}) : F(GY) \to Y$$

In other words $\varepsilon_Y = \overline{\text{id}}_{GY}$.

Given any $\begin{cases} g : FX \to Y & \text{in} \ D \\ f : X \to GY & \text{in} \ C \end{cases}$, by naturality of $\theta$ we have

$$FX \xrightarrow{g} Y \quad \text{and} \quad \varepsilon_Y \circ Ff : FX \xrightarrow{Ff} FGY \xrightarrow{\overline{\text{id}}_{GY}} Y$$

$$X \xrightarrow{\overline{g}} GY \quad \text{and} \quad F(X) \xrightarrow{f} GY \xrightarrow{\overline{id}_{GY}} GY$$

Hence $g = \varepsilon_Y \circ F \overline{g}$ and $g = \varepsilon_Y \circ Ff \Rightarrow \overline{g} = f$.

Thus we do indeed have (UP).
Proof of the **Theorem**—“if” part:

We are given $F : C \to D$ and for each $Y \in D$ a $C$-object $GY$ and $C$-morphism $\varepsilon_Y : F(GY) \to Y$ satisfying $\text{(UP)}$. We have to

1. **extend** $Y \mapsto GY$ to a functor $G : D \to C$

2. **construct a natural isomorphism**
   $$\theta : \text{Hom}_D \circ (F^{\text{op}} \times \text{id}_D) \cong \text{Hom}_C \circ (\text{id}_{C^{\text{op}}} \times G)$$
Proof of the Theorem—“if” part:

We are given \( F : \mathcal{C} \to \mathcal{D} \) and for each \( Y \in \mathcal{D} \) a \( \mathcal{C} \)-object \( GY \) and \( \mathcal{C} \)-morphism \( \varepsilon_Y : F(GY) \to Y \) satisfying (UP). We have to

1. extend \( Y \mapsto GY \) to a functor \( G : \mathcal{D} \to \mathcal{C} \)

For each \( \mathcal{D} \)-morphism \( g : Y' \to Y \) we get \( F(GY') \xrightarrow{\varepsilon_{Y'}} Y' \xrightarrow{g} Y \) and can apply (UP) to get

\[
Gg \triangleq \overline{g \circ \varepsilon_{Y'}} : GY' \to GY
\]

The uniqueness part of (UP) implies

\[
G \text{id} = \text{id} \quad \text{and} \quad G(g' \circ g) = Gg' \circ Gg
\]

so that we get a functor \( G : \mathcal{D} \to \mathcal{C} \). \( \square \)
Proof of the Theorem—“if” part:

We are given \( F : C \to D \) and for each \( Y \in D \) a \( C \)-object \( GY \) and \( C \)-morphism \( \varepsilon_Y : F(GY) \to Y \) satisfying (UP). We have to

2. construct a natural isomorphism

\[
\theta : \text{Hom}_D \circ (F^{\text{op}} \times \text{id}_D) \cong \text{Hom}_C \circ (\text{id}_{C^{\text{op}}} \times G)
\]

Since for all \( g : FX \to Y \) there is a unique \( f : X \to GY \) with \( g = \varepsilon_Y \circ Ff \),

\[
f \mapsto \overline{f} \triangleq \varepsilon_Y \circ Ff
\]

determines a bijection \( C(X, GY) \cong C(FX, Y) \); and it is natural in \( X \) & \( Y \) because

\[
\overline{Gv \circ f \circ u} \triangleq \varepsilon_{Y'} \circ F(Gv \circ f \circ u)
\]

\[
= (\varepsilon_Y \circ F(Gv)) \circ Ff \circ Fu \quad \text{since } F \text{ is a functor}
\]

\[
= (v \circ \varepsilon_Y) \circ Ff \circ Fu \quad \text{by definition of } Gv
\]

\[
= v \circ \overline{f} \circ Fu \quad \text{by definition of } \overline{f}
\]

So we can take \( \theta \) to be the inverse of this natural isomorphism. \( \square \)
Dual of the Theorem:

\( G : C \leftarrow D \) has a left adjoint iff for all \( X \in C \) there are \( FX \in D \) and \( \eta_X \in C(X, G(FX)) \) with the universal property:

for all \( Y \in D \) and \( f \in C(X, GY) \) there is a unique \( \overline{f} \in D(FX, Y) \) satisfying \( G\overline{f} \circ \eta_X = f \)
**Dual of the Theorem:**

$G : C \leftarrow D$ has a left adjoint iff for all $X \in C$ there are $FX \in D$ and $\eta_X \in C(X, G(FX))$ with the universal property:

for all $Y \in D$ and $f \in C(X, GY)$ there is a unique $\overline{f} \in D(FX, Y)$ satisfying $G\overline{f} \circ \eta_X = f$

E.g. we can conclude that the forgetful functor $U : \text{Mon} \rightarrow \text{Set}$ has a left adjoint $F : \text{Set} \rightarrow \text{Mon}$, because of the universal property of

$$F\Sigma \triangleq (\text{List} \Sigma, @, \text{nil}) \quad \text{and} \quad \eta_\Sigma : \Sigma \rightarrow \text{List} \Sigma$$

noted in Lecture 3.
Why are adjoint functors important/useful?

Their universal property (UP) usually embodies some useful mathematical construction

(e.g. “freely generated structures are left adjoints for forgetting-structure”)

and pins it down uniquely up to isomorphism.