Lecture 10

Exercise Sheet 4
(graded, 25% of final course mark)

RETURN SOLUTIONS TO
THE GRADUATE EDUCATION OFFICE
BY 16:00 ON TUESDAY 19 November
GENERAL THEORY OF NATURAL EQUIVALENCES

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Appendix. Representations of categories

Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space $L$ and its "dual".

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Functors
are the appropriate notion of morphism between categories

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C} \to \mathbf{D}$ is specified by:

1. a function $\text{obj } \mathbf{C} \to \text{obj } \mathbf{D}$ whose value at $X$ is written $FX$

2. for all $X, Y \in \mathbf{C}$, a function $\mathbf{C}(X, Y) \to \mathbf{D}(FX, FY)$ whose value at $f : X \to Y$ is written $Ff : FX \to FY$

and which is required to preserve composition and identity morphisms:

$$F(g \circ f) = Fg \circ Ff$$
$$F(\text{id}_X) = \text{id}_{FX}$$
Examples of functors

“Forgetful” functors from categories of set-with-structure back to Set.

E.g. \( U : \text{Mon} \to \text{Set} \)

\[
\begin{align*}
U(M, \cdot, e) & = M \\
U((M_1, \cdot_1, e_1) \xrightarrow{f} (M_2, \cdot_2, e_2)) & = M_1 \xrightarrow{f} M_2
\end{align*}
\]
Examples of functors

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\end{align*}
\]

Similarly $U : \text{Preord} \to \text{Set}$. 
Examples of functors

Free monoid functor \( F : \text{Set} \to \text{Mon} \)

Given \( \Sigma \in \text{Set} \),

\[
F \Sigma = (\text{List} \Sigma, @, \text{nil}), \text{ the free monoid on } \Sigma
\]
Examples of functors

Free monoid functor \( F : \text{Set} \rightarrow \text{Mon} \)

Given \( \Sigma \in \text{Set} \),

\[
F \Sigma = (\text{List } \Sigma, @, \text{nil}), \text{ the free monoid on } \Sigma
\]

Given a function \( f : \Sigma_1 \rightarrow \Sigma_2 \), we get a function

\( Ff : \text{List } \Sigma_1 \rightarrow \text{List } \Sigma_2 \) by mapping \( f \) over finite lists:

\[
Ff [a_1, \ldots, a_n] = [f a_1, \ldots, f a_n]
\]

This gives a monoid morphism \( F \Sigma_1 \rightarrow F \Sigma_2 \); and mapping over lists preserves composition \( (F(g \circ f) = Fg \circ Ff) \) and identities \( (F \text{id}_\Sigma = \text{id}_{F \Sigma}) \). So we do get a functor from \( \text{Set} \) to \( \text{Mon} \).
Examples of functors

If $C$ is a category with binary products and $X \in C$, then the function $(\_ \times X) : \text{obj } C \rightarrow \text{obj } C$ extends to a functor $(\_ \times X) : C \rightarrow C$ mapping morphisms $f : Y \rightarrow Y'$ to

$$f \times \text{id}_X : Y \times X \rightarrow Y' \times X$$

(recall that $f \times g$ is the unique morphism with

$$\begin{align*}
\text{fst} \circ (f \times g) &= f \circ \text{fst} \\
\text{snd} \circ (f \times g) &= g \circ \text{snd}
\end{align*}$$

since it is the case that

$$\begin{align*}
\text{id}_X \times \text{id}_Y &= \text{id}_{X \times Y} \\
(f' \circ f) \times \text{id}_X &= (f' \times \text{id}_X) \circ (f \times \text{id}_X)
\end{align*}$$

(see Exercise Sheet 2, question 1c).
Examples of functors

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $(\_)^X : \text{obj } \mathbf{C} \to \text{obj } \mathbf{C}$ extends to a functor $(\_)^X : \mathbf{C} \to \mathbf{C}$ mapping morphisms $f : Y \to Y'$ to

$$f^X \triangleq \text{cur}(f \circ \text{app}) : Y^X \to Y'^X$$

since it is the case that

$$\begin{cases} (\text{id}_Y)^X = \text{id}_{Y^X} \\ (g \circ f)^X = g^X \circ f^X \end{cases}$$

(see Exercise Sheet 3, question 4).
Contravariance

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C}^{\text{op}} \to \mathbf{D}$ is called a \textit{contravariant functor} from $\mathbf{C}$ to $\mathbf{D}$.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$, then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbf{C}^{\text{op}}$

so $FX \xleftarrow{Ff} FY \xleftarrow{Fg} FZ$ in $\mathbf{D}$ and hence

\[ F(g \circ_{\mathbf{C}} f) = Ff \circ_{\mathbf{D}} Fg \]

(contravariant functors reverse the order of composition)

A functor $\mathbf{C} \to \mathbf{D}$ is sometimes called a \textit{covariant functor} from $\mathbf{C}$ to $\mathbf{D}$. 
Example of a contravariant functor

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $X(-) : \text{obj} \mathbf{C} \to \text{obj} \mathbf{C}$ extends to a functor $X(-) : \mathbf{C}^{\text{op}} \to \mathbf{C}$ mapping morphisms $f : Y \to Y'$ to

$$X^f \triangleq \text{cur(app} \circ (\text{id}_{X^{Y'}} \times f)) : X^{Y'} \to X^Y$$

since it is the case that

$$\begin{align*}
X^{\text{id}_Y} &= \text{id}_{X^Y} \\
X^{g \circ f} &= X^f \circ X^g
\end{align*}$$

(see Exercise Sheet 3, question 5).
Example of a contravariant functor

If \( C \) is a cartesian closed category and \( X \in C \), then the function \( X(-) : \text{obj } C \to \text{obj } C \) extends to a functor \( X(-) : C^{\text{op}} \to C \) mapping morphisms \( f : Y \to Y' \) to

\[
Xf \triangleq \text{cur}(\text{app} \circ (\text{id}_{XY'} \times f)) : X^{Y'} \to X^Y
\]

since it is the case that

\[
\begin{cases}
X^{\text{id}_Y} = \text{id}_{XY} \\
X^{g \circ f} = X^f \circ X^g
\end{cases}
\]

(see Exercise Sheet 3, question 5).

[Non-example of a functor \( \text{Set} \to \text{Set} \)]

\[
X \in \text{Set} \mapsto X \times X \in \text{Set}
\]
Note that since a functor $F : \mathcal{C} \to \mathcal{D}$ preserves domains, codomains, composition and identity morphisms it sends commutative diagrams in $\mathcal{C}$ to commutative diagrams in $\mathcal{D}$

E.g.

\[
\begin{array}{ccc}
\begin{array}{ccc}
Y & \rightarrow & X \\
\downarrow g & & \downarrow h \\
Z & \leftarrow & \\
\end{array} & \rightarrow & \\
\begin{array}{ccc}
F \rightarrow & FX \\
\downarrow Fg \\
FZ \\
\end{array}
\end{array}
\]

$Fh = F(g \circ f) = Fg \circ Ff$
Note that since a functor $F : \mathbf{C} \to \mathbf{D}$ preserves domains, codomains, composition and identity morphisms
it sends isomorphisms in $\mathbf{C}$ to isomorphisms in $\mathbf{D}$, because

$$F(f^{-1}) = (Ff)^{-1}$$
Composing functors

Given functors $F : C \rightarrow D$ and $G : D \rightarrow E$, we get a functor $G \circ F : C \rightarrow E$ with

$$G \circ F \left( \begin{array}{c} X \\ f \\ Y \end{array} \right) = \begin{array}{c} G(FX) \\ G(Ff) \\ G(FY) \end{array}$$

(this preserves composition and identity morphisms, because $F$ and $G$ do)
Identity functor

on a category \( \mathcal{C} \) is \( \text{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C} \) where

\[
\text{id}_\mathcal{C} \begin{pmatrix} X \\ f \\ Y \end{pmatrix} = \begin{pmatrix} X \\ f \\ Y \end{pmatrix}
\]
Functor composition and identity functors satisfy

associativity \[ H \circ (G \circ F) = (H \circ G) \circ F \]
unity \[ \text{id}_D \circ F = F = F \circ \text{id}_C \]

So we can get categories whose objects are categories and whose morphisms are functors
but we have to be a bit careful about size...
One of the axioms of set theory is that set membership is a well-founded relation, that is, there is no infinite sequence of sets $X_0, X_1, X_2, \ldots$ with

$$\cdots \in X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$$

So in particular there is no set $X$ with $X \in X$.

So we cannot form the “set of all sets” or the “category of all categories”.
One of the axioms of set theory is

**set membership is a well-founded relation**, that is, there is no infinite sequence of sets $X_0, X_1, X_2, \ldots$ with

$$\cdots \in X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$$

So in particular there is no set $X$ with $X \in X$.

So we cannot form the “set of all sets” or the “category of all categories”.

But we do assume there are (lots of) big sets

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$

where “big” means each $\mathcal{U}_n$ is a Grothendieck universe…
Grothendieck universes

A Grothendieck universe $\mathcal{U}$ is a set of sets satisfying

- $X \in Y \in \mathcal{U} \Rightarrow X \in \mathcal{U}$
- $X, Y \in \mathcal{U} \Rightarrow \{X, Y\} \in \mathcal{U}$
- $X \in \mathcal{U} \Rightarrow \mathcal{P}X \triangleq \{Y \mid Y \subseteq X\} \in \mathcal{U}$
- $X \in \mathcal{U} \land F \in \mathcal{U}^X \Rightarrow$
  \[
  \{y \mid \exists x \in X, \ y \in Fx\} \in \mathcal{U}
  \]
  (hence also $X, Y \in \mathcal{U} \Rightarrow X \times Y \in \mathcal{U} \land Y^X \in \mathcal{U}$)

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

- $\mathbb{N} \in \mathcal{U}$
We assume there is an infinite sequence \( U_0 \in U_1 \in U_2 \in \cdots \) of bigger and bigger Grothendieck universes

and revise the previous definition of “the” category of sets and functions:

\[
\text{Set}_n = \text{category whose objects are all the sets in } U_n \text{ and with } \text{Set}_n(X, Y) = Y^X = \text{all functions from } X \text{ to } Y.
\]

**Notation:** \( \text{Set} \triangleq \text{Set}_0 \) — its objects are called small sets (and other sets we call large).
Size

**Set** is the category of small sets.

**Definition.** A category $\mathbf{C}$ is **locally small** if for all $X, Y \in \mathbf{C}$, the set of $\mathbf{C}$-morphisms $X \to Y$ is small, that is, $\mathbf{C}(X, Y) \in \mathbf{Set}$.

$\mathbf{C}$ is a **small category** if it is both locally small and $\text{obj } \mathbf{C} \in \mathbf{Set}$.

E.g. **Set**, **Preord** and **Mon** are all locally small (but not small).

Given $P \in \text{Preord}$, the category $\mathbf{C}_P$ it determines is small; similarly, the category $\mathbf{C}_M$ determined by $M \in \text{Mon}$ is small.
The category of small categories, \textbf{Cat}

- objects are all small categories
- morphisms in $\textbf{Cat}(C, D)$ are all functors $C \rightarrow D$
- composition and identity morphisms as for functors

\textbf{Cat} is a locally small category