Lecture 7

IPL entailment $\boldsymbol{\Phi \vdash \varphi}$

Recall the rules:

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\mathrm{AX}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\mathrm{WK}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\mathrm{CUT})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\mathrm{TRUE}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\mathrm{I}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\mathrm{I})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\mathrm{E}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\mathrm{E}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\mathrm{E})$$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\cdots}\,(\mathrm{AX})
    }{
      \Phi, \varphi \vdash \psi \Rightarrow \theta
    }\,(\mathrm{WK})
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\cdots}\,(\mathrm{AX})
        }{\cdots}\,(\mathrm{WK})
      }{
        \Phi, \varphi \vdash \varphi \Rightarrow \psi
      }\,(\mathrm{WK})
      \qquad
      \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\mathrm{AX})
    }{
      \Phi, \varphi \vdash \psi
    }\,(\Rightarrow\mathrm{E})
  }{
    \Phi, \varphi \vdash \theta
  }\,(\Rightarrow\mathrm{E})
}{
  \Phi \vdash \varphi \Rightarrow \theta
}\,(\Rightarrow\mathrm{I})
$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\cdots}\,(\mathrm{AX})
        }{\cdots}\,(\mathrm{WK})
      }{
        \Psi \vdash \varphi \Rightarrow \psi
      }\,(\mathrm{WK})
      \qquad
      \dfrac{}{\Psi \vdash \varphi}\,(\mathrm{AX})
    }{
      \Psi \vdash \psi
    }\,(\Rightarrow\mathrm{E})
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\cdots}\,(\mathrm{AX})
        }{\cdots}\,(\mathrm{WK})
      }{
        \Psi, \psi \vdash \psi \Rightarrow \theta
      }\,(\mathrm{WK})
      \qquad
      \dfrac{}{\Psi, \psi \vdash \psi}\,(\mathrm{AX})
    }{
      \Psi, \psi \vdash \theta
    }\,(\Rightarrow\mathrm{E})
  }{
    \Psi \vdash \theta
  }\,(\mathrm{CUT})
}{
  \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta
}\,(\Rightarrow\mathrm{I})
$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi,\varphi \vdash \psi \Rightarrow \theta}\,(\text{WK}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi,\varphi \vdash \varphi \Rightarrow \psi}\,(\text{WK}) \quad \dfrac{}{\Phi,\varphi \vdash \varphi}\,(\text{AX})}{\Phi,\varphi \vdash \psi}\,(\Rightarrow\text{E})}{\Phi,\varphi \vdash \theta}\,(\Rightarrow\text{E})}{\Phi \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Psi \vdash \varphi \Rightarrow \psi}\,(\text{WK}) \quad \dfrac{}{\Psi \vdash \varphi}\,(\text{AX})}{\Psi \vdash \psi}\,(\Rightarrow\text{E}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Psi,\psi \vdash \psi \Rightarrow \theta}\,(\text{WK}) \quad \dfrac{}{\Psi,\psi \vdash \psi}\,(\text{AX})}{\Psi,\psi \vdash \theta}\,(\Rightarrow\text{E})}{\Psi \vdash \theta}\,(\text{CUT})}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Why is the first proof simpler than the second one?

Proof theory

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \cancel{\frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})}$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\mathrm{I}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ (\Rightarrow\mathrm{I})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\mathrm{E}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\mathrm{E}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ (\Rightarrow\mathrm{E})$$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the $(\textsc{cut})$ rule.

Proof theory

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\mathrm{I}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\mathrm{I})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\mathrm{E}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\mathrm{E}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\mathrm{E})$$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the $(\textsc{cut})$ rule.

Simply-Typed Lambda Calculus provides a language for describing proofs in IPL and their properties. . .

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
`unit`	unit type
$A \times B$	product type
`A -> B`	function type

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
$\texttt{unit}$	unit type
$A \times B$	product type
$A \rightarrow B$	function type

Terms: $s, t, r, \ldots ::=$

c^A	constants (of given type A)
x	variable (countably many)
$()$	unit value
$(s\,,t)$	pair
$\texttt{fst}\,t \quad \texttt{snd}\,t$	projections
$\lambda x : A.\, t$	function abstraction
$s\,t$	function application

STLC

Some examples of terms:

- $\lambda z : (A \to B) \times (A \to C).\, \lambda x : A.\, ((\mathtt{fst}\, z)\, x, (\mathtt{snd}\, z)\, x))$

  (has type $((A \to B) \times (A \to C)) \to (A \to (B \times C)))$)

- $\lambda z : A \to (B \times C).\, (\lambda x : A.\, \mathtt{fst}(z\, x), \lambda y : A.\, \mathtt{snd}(z\, y))$

  (has type $(A \to (B \times C)) \to ((A \to B) \times (A \to C)))$)

- $\lambda z : A \to (B \times C).\, \lambda x : A.\, ((\mathtt{fst}\, z)\, x, (\mathtt{snd}\, z)\, x)$

  (has no type)

STLC typing relation, $\Gamma \vdash t : A$

Γ ranges over typing environments

$$\Gamma ::= \diamond \mid \Gamma, x : A$$

(so typing environments are comma-separated snoc-lists of (variable,type)-pairs
— in fact only the lists whose variables are mutually distinct get used)

The typing relation $\Gamma \vdash t : A$ is inductively defined by the following rules, which make use of the following notation

$\boxed{\Gamma\ \mathtt{ok}}$ means: no variable occurs more than once in Γ

$\boxed{\mathtt{dom}\,\Gamma}$ = finite set of variables occurring in Γ

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for variables

$$\frac{\Gamma\ \mathtt{ok} \qquad x \notin \mathtt{dom}\,\Gamma}{\Gamma, x : A \vdash x : A}\ (\textsc{var})$$

$$\frac{\Gamma \vdash x : A \qquad x' \notin \mathtt{dom}\,\Gamma}{\Gamma, x' : A' \vdash x : A}\ (\textsc{var'})$$

Typing rules for constants and unit value

$$\frac{\Gamma\ \mathtt{ok}}{\Gamma \vdash c^A : A}\ (\textsc{cons})$$

$$\frac{\Gamma\ \mathtt{ok}}{\Gamma \vdash () : \mathtt{unit}}\ (\textsc{unit})$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for pairs and projections

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash (s\,,t) : A \times B}\ (\textsc{pair})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathtt{fst}\, t : A}\ (\textsc{fst})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathtt{snd}\, t : B}\ (\textsc{snd})$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for function abstraction & application

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\, t : A \to B}\ (\textsc{fun})$$

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t : A}{\Gamma \vdash s\, t : B}\ (\textsc{app})$$

STLC typing relation, $\Gamma \vdash t : A$

Example typing derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : B \to C}\,(\textsc{var})}{\Gamma, x : A \vdash g : B \to C}\,(\textsc{var'}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\diamond, f : A \to B \vdash f : A \to B}\,(\textsc{var})}{\Gamma \vdash f : A \to B}\,(\textsc{var'})}{\Gamma, x : A \vdash f : A \to B}\,(\textsc{var'}) \qquad \dfrac{}{\Gamma, x : A \vdash x : A}\,(\textsc{var})}{\Gamma, x : A \vdash f\, x : B}\,(\textsc{app})}{\Gamma, x : A \vdash g(f\, x) : C}\,(\textsc{app})}{\Gamma \vdash \lambda x : A.\, g(f\, x) : A \to C}\,(\textsc{fun})}{\diamond, f : A \to B \vdash \lambda g : B \to C.\, \lambda x : A.\, g(f\, x) : (B \to C) \to (A \to C)}\,(\textsc{fun})}{\diamond \vdash \lambda f : A \to B.\, \lambda g : B \to C.\, \lambda x : A.\, g(f\, x) : (A \to B) \to (B \to C) \to (A \to C)}\,(\textsc{fun})
$$

where $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$

N.B. the STLC typing rules are "syntax-directed", by the structure of terms t and then in the case of variables x, by the structure of typing environments Γ.

Semantics of STLC types in a ccc

Given a cartesian closed category $\mathbf{C}$,

any function M mapping ground types G to objects $M(G) \in \mathbf{C}$

extends to function $A \mapsto M[\![A]\!] \in \mathbf{C}$ and $\Gamma \mapsto M[\![\Gamma]\!] \in \mathbf{C}$ from STLC types and typing environments to $\mathbf{C}$-objects, by recursion on the structure of A:

$$
\begin{aligned}
M[\![G]\!] &= M(G) && \\
M[\![\mathtt{unit}]\!] &= 1 && \text{terminal object in } \mathbf{C} \\
M[\![A \times B]\!] &= M[\![A]\!] \times M[\![B]\!] && \text{product in } \mathbf{C} \\
M[\![A \to B]\!] &= M[\![A]\!] \to M[\![B]\!] && \text{exponential in } \mathbf{C} \\
\\
M[\![\diamond]\!] &= 1 && \text{terminal object in } \mathbf{C} \\
M[\![\Gamma, x : A]\!] &= M[\![\Gamma]\!] \times M[\![A]\!] && \text{product in } \mathbf{C}
\end{aligned}
$$

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
  (which extends to a function mapping all types to objects, $A \mapsto M[\![A]\!]$, as we have seen)

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \to M[\![A]\!]$
  (In a category with a terminal object 1, given an object $X \in \mathbf{C}$, morphisms $1 \to X$ are sometimes called global elements of X.)

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \to M[\![A]\!]$

we get a function mapping provable instances of the typing relation $\Gamma \vdash t : A$ to $\mathbf{C}$-morphisms

$$\boxed{M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \to M[\![A]\!]}$$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules (which follows the structure of t):

Semantics of STLC terms in a ccc

Variables:

$$M[\![\Gamma, x : A \vdash x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!] \xrightarrow{\pi_2} M[\![A]\!]$$

$$M[\![\Gamma, x' : A' \vdash x : A]\!] =$$

$$M[\![\Gamma]\!] \times M[\![A']\!] \xrightarrow{\pi_1} M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash x:A]\!]} M[\![A]\!]$$

Constants:

$$M[\![\Gamma \vdash c^A : A]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c^A)} M[\![A]\!]$$

Unit value:

$$M[\![\Gamma \vdash () : \mathtt{unit}]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1$$