

Lecture 4

Binary products

In a category \mathbf{C} , a **product** for objects $X, Y \in \mathbf{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ with the universal property:

For all $X \xleftarrow{f} Z \xrightarrow{g} Y$ in \mathbf{C} , there is a unique \mathbf{C} -morphism $h : Z \rightarrow P$ such that the following diagram commutes in \mathbf{C} :

$$\begin{array}{ccccc} & & \mathbf{Z} & & \\ & f \swarrow & \downarrow h & \searrow g & \\ \mathbf{X} & \xleftarrow{\pi_1} & \mathbf{P} & \xrightarrow{\pi_2} & \mathbf{Y} \end{array}$$

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So (P, π_1, π_2) is a terminal object in the category with

- ▶ objects: (Z, f, g) where $X \xleftarrow{f} Z \xrightarrow{g} Y$ in \mathbf{C}
- ▶ morphisms $h : (Z_1, f_1, g_1) \rightarrow (Z_2, f_2, g_2)$ are $h \in \mathbf{C}(Z_1, Z_2)$ such that $f_1 = f_2 \circ h$ and $g_1 = g_2 \circ h$
- ▶ composition and identities as in \mathbf{C}

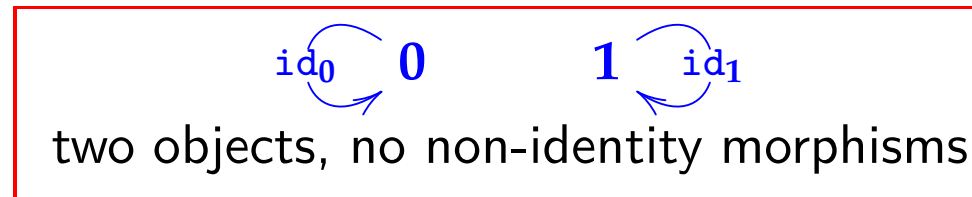
So if it exists, the binary product of two objects in a category is unique up to (unique) isomorphism.

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N.B. products of objects in a category do not always exist. For example in the category



the objects 0 and 1 do not have a product, because there is no diagram of the form $0 \xleftarrow{?} \rightarrow 1$ in this category.

Notation for binary products

Assuming \mathbf{C} has binary products of objects, the product of $X, Y \in \mathbf{C}$ is written

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

and given $X \xleftarrow{f} Z \xrightarrow{g} Y$, the unique $h : Z \rightarrow X \times Y$ with $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ is written

$$\langle f, g \rangle : Z \rightarrow X \times Y$$

Examples of products

In **Set**, category-theoretic products are given by the usual cartesian product of sets (set of all ordered pairs)

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$

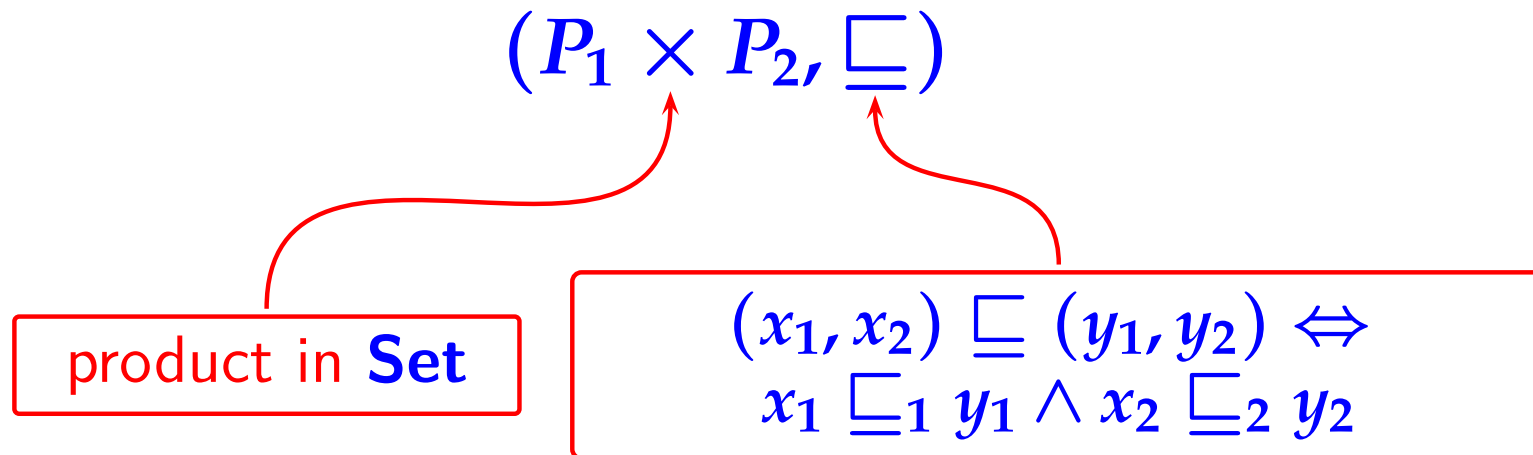
$$\pi_1(x, y) = x$$

$$\pi_2(x, y) = y$$

because...

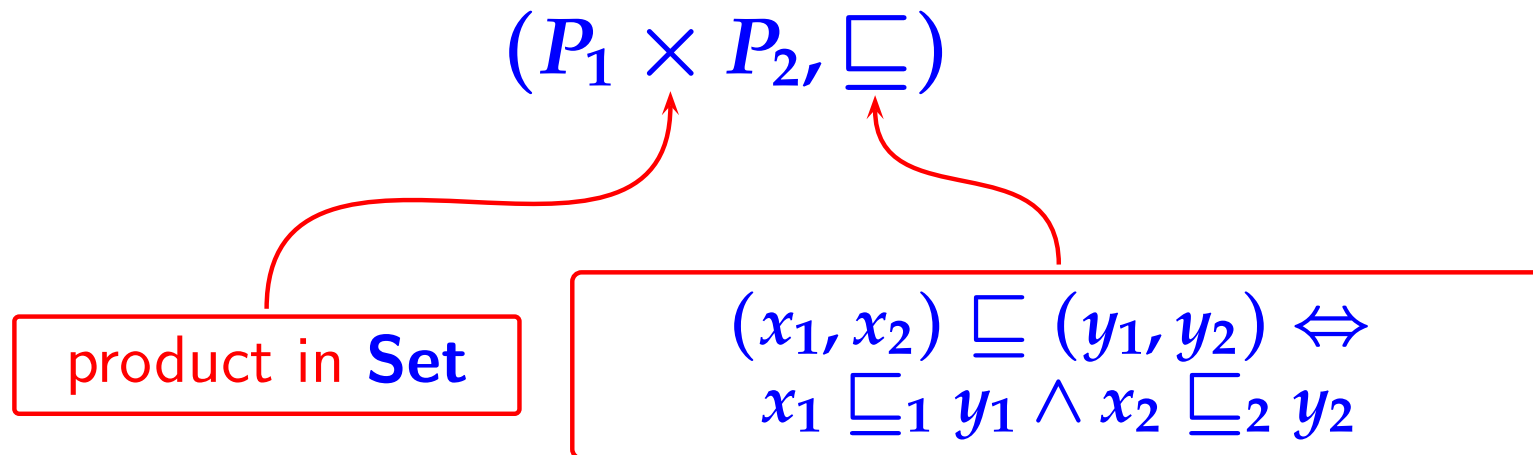
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The projection functions $P_1 \xleftarrow{\pi_1} P_1 \times P_2 \xrightarrow{\pi_2} P_2$ are monotone for this pre-order on $P_1 \times P_2$ and have the universal property needed for a product in **Preord** (check).

Examples of products

In **Mon**, can take product of (M_1, \cdot_1, e_1) and (M_2, \cdot_2, e_2) to be

$$(M_1 \times M_2, \cdot, (e_1, e_2))$$

product in **Set**

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot_1 y_1, x_2 \cdot_2 y_2)$$

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The projection functions $M_1 \xleftarrow{\pi_1} M_1 \times M_2 \xrightarrow{\pi_2} M_2$ are monoid morphisms for this monoid structure on $M_1 \times M_2$ and have the universal property needed for a product in **Mon** (check).

Examples of products

Recall that each pre-ordered set (P, \sqsubseteq) determines a category \mathbf{C}_P .

Given $p, q \in P = \text{obj } \mathbf{C}_P$, the product $p \times q$ (if it exists) is a **greatest lower bound** (or **glb**, or **meet**) for p and q in (P, \sqsubseteq) :

lower bound:

$$p \times q \sqsubseteq p \wedge p \times q \sqsubseteq q$$

greatest among all lower bounds:

$$\forall r \in P, r \sqsubseteq p \wedge r \sqsubseteq q \Rightarrow r \sqsubseteq p \times q$$

Notation: glbs are often written $p \wedge q$ or $p \sqcap q$

Duality

A binary **coproduct** of two objects in a category \mathbf{C} is their product in the category \mathbf{C}^{op} .

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A binary **coproduct** of two objects in a category **C** is their product in the category **C^{op}**.

Thus the coproduct of $X, Y \in \mathbf{C}$
if it exists,

is a diagram $X \xrightarrow{\text{inl}} X + Y \xleftarrow{\text{inr}} Y$
with the universal property:

$$\forall (X \xrightarrow{f} Z \xleftarrow{g} Y),$$

$$\exists! (X + Y \xrightarrow{h} Z),$$

$$f = h \circ \text{inl} \wedge g = h \circ \text{inr}$$

Duality

A binary **coproduct** of two objects in a category **C** is their product in the category **C^{op}**.

E.g. in **Set**, the coproduct of **X** and **Y**

$$X \xrightarrow{\text{inl}} X + Y \xleftarrow{\text{inr}} Y$$

is given by their **disjoint union** (tagged sum)

$$\begin{aligned} X + Y &= \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\} \\ \text{inl}(x) &= (0, x) \\ \text{inr}(y) &= (1, y) \end{aligned}$$

(prove this)