

Lecture 2

Recall

A **category** \mathbf{C} is specified by

- ▶ a set $\text{obj } \mathbf{C}$ whose elements are called **C-objects**
- ▶ for each $X, Y \in \text{obj } \mathbf{C}$, a set $\mathbf{C}(X, Y)$ whose elements are called **C-morphisms from X to Y**
- ▶ a function assigning to each $X \in \text{obj } \mathbf{C}$ an element $\text{id}_X \in \mathbf{C}(X, X)$ called the **identity morphism** for the **C-object** X
- ▶ a function assigning to each $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ (where $X, Y, Z \in \text{obj } \mathbf{C}$) an element $g \circ f \in \mathbf{C}(X, Z)$ called the **composition** of **C-morphisms** f and g and satisfying **associativity** and **unity** properties.

Example: category of pre-orders, **Preord**

- ▶ objects are sets P equipped with a **pre-order** $_ \sqsubseteq _$
— a binary relation on P that is
reflexive: $\forall x \in P, x \sqsubseteq x$
transitive: $\forall x, y, z \in P, x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$

A **partial order** is a pre-order that is also

anti-symmetric: $\forall x, y \in P, x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

Example: category of pre-orders, **Preord**

- ▶ objects are sets P equipped with a pre-order \sqsubseteq
- ▶ morphisms: $\mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{f \in \mathbf{Set}(P_1, P_2) \mid f \text{ is monotone}\}$

$$\forall x, x' \in P_1, x \sqsubseteq_1 x' \Rightarrow f x \sqsubseteq_2 f x'$$

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- ▶ identities and composition: as for **Set**

Q: why is this well-defined?

A: because the set of monotone functions contains identity functions and is closed under composition.

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Pre- and partial orders are relevant to the denotational semantics of programming languages (among other things).

Example: category of monoids, **Mon**

- ▶ objects are **monoids** (M, \cdot, e) — set M equipped with a binary operation $_ \cdot _ : M \times M \rightarrow M$ which is

associative $\forall x, y, z \in M, x \cdot (y \cdot z) = (x \cdot y) \cdot z$
has e as its unit $\forall x \in M, e \cdot x = x = x \cdot e$

CS-relevant example of a monoid: $(\text{List } \Sigma, @, \text{nil})$ where

$\text{List } \Sigma$ = set of finite lists of elements of set Σ
 $@$ = list concatenation
 $\text{nil} @ l = l$
 $(a :: l) @ l' = a :: (l @ l')$
 nil = empty list

Example: category of monoids, **Mon**

- ▶ objects are **monoids** (M, \cdot, e)
- ▶ morphisms: $\mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{f \in \mathbf{Set}(M_1, M_2) \mid f e_1 = e_2 \wedge \forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y)\}$

It's common to denote a monoid (M, \cdot, e) just by its underlying set M , leaving $_ \cdot _$ and e implicit (hence the same notation gets used for different instances of monoid operations).

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- ▶ identities and composition: as for **Set**

Q: why is this well-defined?

A: because the set of functions that are monoid morphisms contains identity functions and is closed under composition.

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 $\quad \forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y)\}$
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Monoids are relevant to **automata theory** (among other things).

Example: each pre-order determines a category

Given a pre-ordered set (P, \sqsubseteq) ,
we get a category \mathbf{C}_P by taking

▶ objects $\text{obj } \mathbf{C}_P = P$

▶ morphisms $\mathbf{C}_P(x, y) \triangleq \begin{cases} \mathbf{1} & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$

(where $\mathbf{1}$ is some fixed one-element set and \emptyset is the empty set)

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▶ identity morphisms and composition are uniquely determined (why?)

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- ▶ identity morphisms and composition are uniquely determined (why?)

E.g. when (P, \sqsubseteq) has just one element 0

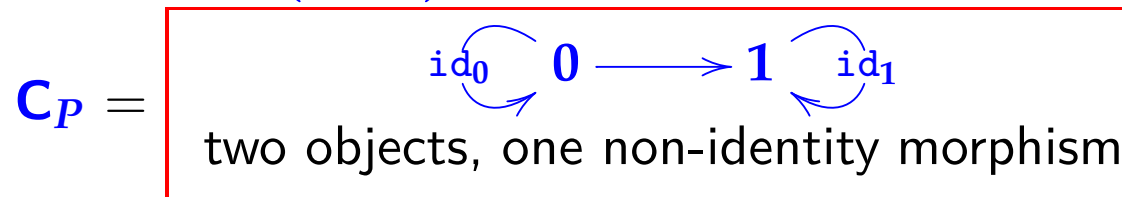
$$\mathbf{C}_P = \boxed{\begin{array}{c} 0 \begin{array}{c} \curvearrowright \text{id}_0 \\ \curvearrowleft \end{array} \\ \text{one object, one morphism} \end{array}}$$

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E.g. when (P, \sqsubseteq) has just two elements $0 \sqsubseteq 1$

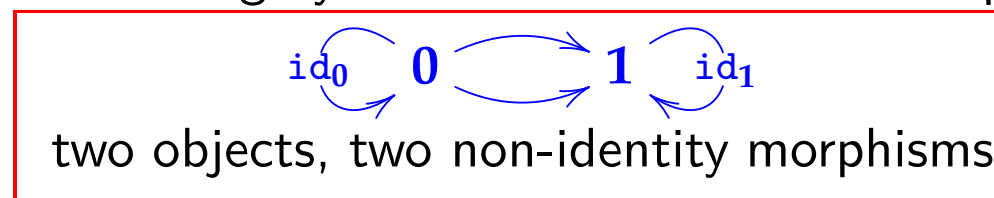


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- ▶ identity morphisms and composition are uniquely determined (why?)

Example of a finite category that does not arise from a pre-ordered set:



Example: each monoid determines a category

Given a monoid (M, \cdot, e) ,
we get a category \mathbf{C}_M by taking

- ▶ objects: $\text{obj } \mathbf{C}_M = \mathbf{1} = \{0\}$ (one-element set)
- ▶ morphisms: $\mathbf{C}_M(0, 0) = M$
- ▶ identity morphism: $\text{id}_0 = e \in M = \mathbf{C}_M(0, 0)$
- ▶ composition of $f \in \mathbf{C}_M(0, 0)$ and $g \in \mathbf{C}_M(0, 0)$ is $g \cdot f \in M = \mathbf{C}_M(0, 0)$

Definition of isomorphism

Let \mathbf{C} be a category. A \mathbf{C} -morphism $f : X \rightarrow Y$ is an **isomorphism** if there is some $g : Y \rightarrow X$ for which

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \text{id}_X & \downarrow g \\ & & X & \xrightarrow{f} & Y \\ & & & \nearrow \text{id}_Y & \end{array}$$

is a commutative diagram.

Definition of isomorphism

Let \mathbf{C} be a category. A \mathbf{C} -morphism $f : X \rightarrow Y$ is an **isomorphism** if there is some $g : Y \rightarrow X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

▶ Such a g is uniquely determined by f (why?) and we write f^{-1} for it.

▶ Given $X, Y \in \mathbf{C}$, if such an f exists, we say the objects X and Y are **isomorphic** in \mathbf{C} and write

$$X \cong Y$$

(There may be many different f that witness the fact that X and Y are isomorphic.)

Theorem. A function $f \in \mathbf{Set}(X, Y)$ is an isomorphism in the category **Set** iff f is a bijection, that is

▶ **injective:** $\forall x, x' \in X, f x = f x' \Rightarrow x = x'$

▶ **surjective:** $\forall y \in Y, \exists x \in X, f x = y$

Proof...

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Proof...

Theorem. A monoid morphism $f \in \mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2))$ is an isomorphism in the category **Mon** iff $f \in \mathbf{Set}(M_1, M_2)$ is a bijection.

Proof...

Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

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Theorem. A monotone function $f \in \mathbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category **Poset** iff $f \in \mathbf{Set}(P_1, P_2)$ is a surjective function satisfying

► **reflective**: $\forall x, x' \in P_1, f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'$

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Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

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Proof...

(Why does this characterisation not work for **Preord**?)