Lecture 2

Recall

A category $\mathbf{C}$ is specified by

- a set $\boxed{\mathtt{obj}\,\mathbf{C}}$ whose elements are called $\mathbf{C}$-objects
- for each $X, Y \in \mathtt{obj}\,\mathbf{C}$, a set $\boxed{\mathbf{C}(X,Y)}$ whose elements are called $\mathbf{C}$-morphisms from X to Y
- a function assigning to each $X \in \mathtt{obj}\,\mathbf{C}$ an element $\boxed{\mathtt{id}_X \in \mathbf{C}(X,X)}$ called the identity morphism for the $\mathbf{C}$-object X
- a function assigning to each $f \in \mathbf{C}(X,Y)$ and $g \in \mathbf{C}(Y,Z)$ (where $X, Y, Z \in \mathtt{obj}\,\mathbf{C}$) an element $\boxed{g \circ f \in \mathbf{C}(X,Z)}$ called the composition of $\mathbf{C}$-morphisms f and g and satisfying associativity and unity properties.

Example:
category of pre-orders, **Preord**

- objects are sets P equipped with a pre-order $\_ \sqsubseteq \_$ — a binary relation on P that is
  reflexive: $\forall x \in P,\ x \sqsubseteq x$
  transitive: $\forall x, y, z \in P,\ x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$

A partial order is a pre-order that is also
anti-symmetric: $\forall x, y \in P,\ x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

Example: category of pre-orders, **Preord**

- objects are sets P equipped with a pre-order $\_ \sqsubseteq \_$
- morphisms: $\mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{f \in \mathbf{Set}(P_1, P_2) \mid f \text{ is monotone}\}$

$$\boxed{\forall x, x' \in P_1,\ x \sqsubseteq_1 x' \Rightarrow f\,x \sqsubseteq_2 f\,x'}$$

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- identities and composition: as for **Set**

Q: why is this well-defined?

A: because the set of monotone functions contains identity functions and is closed under composition.

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Pre- and partial orders are relevant to the denotational semantics of programming languages (among other things).

Example: category of monoids, **Mon**

- objects are monoids $(M, \cdot, e)$ — set M equipped with a binary operation $\_ \cdot \_ : M \times M \to M$ which is
  associative $\forall x, y, z \in M,\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$
  has e as its unit $\forall x \in M,\ e \cdot x = x = x \cdot e$

  CS-relevant example of a monoid: $(\mathtt{List}\,\Sigma, @, \mathtt{nil})$ where

$$
\begin{array}{rcl}
\mathtt{List}\,\Sigma & = & \text{set of finite lists of elements of set } \Sigma \\
@ & = & \text{list concatenation} \\
 & & \quad \mathtt{nil} \mathbin{@} \ell = \ell \\
 & & \quad (a :: \ell) \mathbin{@} \ell' = a :: (\ell \mathbin{@} \ell') \\
\mathtt{nil} & = & \text{empty list}
\end{array}
$$

Example:
category of monoids, **Mon**

- objects are monoids $(M, \cdot, e)$
- morphisms: $\mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq$
$\{f \in \mathbf{Set}(M_1, M_2) \mid f\, e_1 = e_2 \wedge$
$\quad \forall x, y \in M_1,\ f(x \cdot_1 y) = (f\, x) \cdot_2 (f\, y)\}$

It's common to denote a monoid $(M, \cdot, e)$ just by its underlying set M, leaving $\_ \cdot \_$ and e implicit (hence the same notation gets used for different instances of monoid operations).

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category of monoids, **Mon**

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- morphisms: $\mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq$
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- identities and composition: as for **Set**

Q: why is this well-defined?
A: because the set of functions that are monoid morphisms contains identity functions and is closed under composition.

Example: category of monoids, **Mon**

- objects are monoids $(M, \cdot, e)$
- morphisms: $\mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq$
  $\{f \in \mathbf{Set}(M_1, M_2) \mid f\, e_1 = e_2 \wedge$
  $\forall x, y \in M_1,\ f(x \cdot_1 y) = (f\, x) \cdot_2 (f\, y)\}$
- identities and composition: as for **Set**

Monoids are relevant to automata theory (among other things).

Example: each pre-order determines a category

Given a pre-ordered set $(P, \sqsubseteq)$,
we get a category $\mathbf{C}_P$ by taking

- objects $\texttt{obj}\,\mathbf{C}_P = P$
- morphisms $\mathbf{C}_P(x, y) \triangleq \begin{cases} \mathbf{1} & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$

  (where $\mathbf{1}$ is some fixed one-element set and $\emptyset$ is the empty set)

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- morphisms $\mathbf{C}_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$
- identity morphisms and composition are uniquely determined (why?)

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- identity morphisms and composition are uniquely determined (why?)

E.g. when $(P, \sqsubseteq)$ has just one element $\mathbf{0}$

$$\mathbf{C}_P = \boxed{\begin{array}{c} \mathbf{0} \circlearrowleft \mathtt{id}_{\mathbf{0}} \\ \text{one object, one morphism} \end{array}}$$

Example: each pre-order determines a category

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- morphisms $\mathbf{C}_P(x, y) \triangleq \begin{cases} \mathbf{1} & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$
- identity morphisms and composition are uniquely determined (why?)

E.g. when $(P, \sqsubseteq)$ has just two elements $\mathbf{0} \sqsubseteq \mathbf{1}$

$$\mathbf{C}_P = \boxed{\begin{tikzcd} \mathbf{0} \arrow[loop left, "\mathtt{id}_0"] \arrow[r] & \mathbf{1} \arrow[loop right, "\mathtt{id}_1"] \end{tikzcd}}$$

two objects, one non-identity morphism

Example: each pre-order determines a category

Given a pre-ordered set $(P, \sqsubseteq)$,
we get a category $\mathbf{C}_P$ by taking

- objects $\mathtt{obj}\,\mathbf{C}_P = P$
- morphisms $\mathbf{C}_P(x, y) \triangleq \begin{cases} \mathbf{1} & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}$
- identity morphisms and composition are uniquely determined (why?)

Example of a finite category that does not arise from a pre-ordered set:

$$\begin{tikzcd} \mathbf{0} \arrow[loop left, "\mathtt{id}_0"] \arrow[r, bend left] \arrow[r, bend right] & \mathbf{1} \arrow[loop right, "\mathtt{id}_1"] \end{tikzcd}$$

two objects, two non-identity morphisms

Example: each monoid determines a category

Given a monoid $(M, \cdot, e)$,
we get a category $\mathbf{C}_M$ by taking

- objects: $\mathtt{obj}\,\mathbf{C}_M = 1 = \{0\}$ (one-element set)
- morphisms: $\mathbf{C}_M(0,0) = M$
- identity morphism: $\mathtt{id}_0 = e \in M = \mathbf{C}_M(0,0)$
- composition of $f \in \mathbf{C}_M(0,0)$ and $g \in \mathbf{C}_M(0,0)$ is $g \cdot f \in M = \mathbf{C}_M(0,0)$

Definition of isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ for which

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathtt{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathtt{id}_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a commutative diagram.

Definition of isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ with $g \circ f = \mathtt{id}_X$ and $f \circ g = \mathtt{id}_Y$.

- Such a g is uniquely determined by f (why?) and we write $\boxed{f^{-1}}$ for it.
- Given $X, Y \in \mathbf{C}$, if such an f exists, we say the objects X and Y are isomorphic in $\mathbf{C}$ and write $\boxed{X \cong Y}$

  (There may be many different f that witness the fact that X and Y are isomorphic.)

Theorem. A function $f \in \mathbf{Set}(X, Y)$ is an isomorphism in the category **Set** iff f is a bijection, that is

- injective: $\forall x, x' \in X,\ f\,x = f\,x' \Rightarrow x = x'$
- surjective: $\forall y \in Y, \exists x \in X,\ f\,x = y$

Proof. . .

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Proof...

Theorem. A monoid morphism $f \in \mathbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2))$ is an isomorphism in the category **Mon** iff $f \in \mathbf{Set}(M_1, M_2)$ is a bijection.

Proof...

Define **Poset** to be the category whose objects are posets $=$ pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

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Theorem. A monotone function $f \in \mathbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category **Poset** iff $f \in \mathbf{Set}(P_1, P_2)$ is a surjective function satisfying

- reflective: $\forall x, x' \in P_1,\ f\,x \sqsubseteq_2 f\,x' \Rightarrow x \sqsubseteq_1 x'$

Proof...

Define **Poset** to be the category whose objects are posets = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

Theorem. A monotone function $f \in \mathbf{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category **Poset** iff $f \in \mathbf{Set}(P_1, P_2)$ is a surjective function satisfying

- reflective: $\forall x, x' \in P_1,\ f\,x \sqsubseteq_2 f\,x' \Rightarrow x \sqsubseteq_1 x'$

Proof...

(Why does this characterisation not work for **Preord**?)