Lecture 2
Recall

A category $\mathbf{C}$ is specified by

- a set $\text{obj } \mathbf{C}$ whose elements are called $\mathbf{C}$-objects
- for each $X, Y \in \text{obj } \mathbf{C}$, a set $\mathbf{C}(X, Y)$ whose elements are called $\mathbf{C}$-morphisms from $X$ to $Y$
- a function assigning to each $X \in \text{obj } \mathbf{C}$ an element $\text{id}_X \in \mathbf{C}(X, X)$ called the identity morphism for the $\mathbf{C}$-object $X$
- a function assigning to each $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ (where $X, Y, Z \in \text{obj } \mathbf{C}$) an element $g \circ f \in \mathbf{C}(X, Z)$ called the composition of $\mathbf{C}$-morphisms $f$ and $g$ and satisfying associativity and unity properties.
Example:
category of pre-orders, \textbf{Preord}

- objects are sets \( P \) equipped with a pre-order \( \sqsubseteq \) — a binary relation on \( P \) that is
  - reflexive: \( \forall x \in P, \ x \sqsubseteq x \)
  - transitive: \( \forall x, y, z \in P, \ x \sqsubseteq y \land y \sqsubseteq z \Rightarrow x \sqsubseteq z \)

A partial order is a pre-order that is also
- anti-symmetric: \( \forall x, y \in P, \ x \subseteq y \land y \subseteq x \Rightarrow x = y \)
Example: category of pre-orders, $\text{Preord}$

- objects are sets $P$ equipped with a pre-order $\sqsubseteq$
- morphisms: $\text{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \}$

\[ \forall x, x' \in P_1, \; x \sqsubseteq_1 x' \Rightarrow f x \sqsubseteq_2 f x' \]
Example: category of pre-orders, $\text{Preord}$

- Objects are sets $P$ equipped with a pre-order $\sqsubseteq$
- Morphisms: $\text{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \}$
- Identities and composition: as for $\text{Set}$

Q: why is this well-defined?
A: because the set of monotone functions contains identity functions and is closed under composition.
Example:
category of pre-orders, \( \text{Preord} \)

- objects are sets \( P \) equipped with a pre-order \( \sqsubseteq \)
- morphisms: \( \text{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \} \)
- identities and composition: as for \( \text{Set} \)

Pre- and partial orders are relevant to the denotational semantics of programming languages (among other things).
Example: category of monoids, \textbf{Mon}

- objects are monoids \((M, \cdot, e)\) — set \(M\) equipped with a binary operation \(\_ \cdot \_ : M \times M \to M\) which is

  associative \(\forall x, y, z \in M, x \cdot (y \cdot z) = (x \cdot y) \cdot z\)

  has \(e\) as its unit \(\forall x \in M, e \cdot x = x = x \cdot e\)

CS-relevant example of a monoid: \(((\text{List} \Sigma, @, \text{nil}))\) where

\[
\begin{align*}
\text{List} \Sigma &= \text{set of finite lists of elements of set } \Sigma \\
@ &= \text{list concatenation} \\
\text{nil} @ \ell &= \ell \\
(a :: \ell) @ \ell' &= a :: (\ell @ \ell') \\
\text{nil} &= \text{empty list}
\end{align*}
\]
Example: category of monoids, \textbf{Mon}

\begin{itemize}
  \item objects are \textbf{monoids} \((M, \cdot, e)\)
  \item morphisms: \textbf{Mon}(((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \\
\{ f \in \textbf{Set}(M_1, M_2) \mid f e_1 = e_2 \land \\
\forall x, y \in M_1, \ f(x \cdot_1 y) = (f x) \cdot_2 (f y) \}\}
\end{itemize}

It's common to denote a monoid \((M, \cdot, e)\) just by its underlying set \(M\),
leaving \_\cdot\_ and \(e\) implicit (hence the same notation gets used for
different instances of monoid operations).
Example: category of monoids, $\textbf{Mon}$

- objects are monoids $(M, \cdot, e)$
- morphisms: $\textbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \textbf{Set}(M_1, M_2) \mid f e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (fx) \cdot_2 (fy) \}$
- identities and composition: as for $\textbf{Set}$

Q: why is this well-defined?
A: because the set of functions that are monoid morphisms contains identity functions and is closed under composition.
Example: category of monoids, $\textbf{Mon}$

- objects are monoids $(M, \cdot, e)$
- morphisms: $\textbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \textbf{Set}(M_1, M_2) \mid f e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y) \}$
- identities and composition: as for $\textbf{Set}$

Monoids are relevant to automata theory (among other things).
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(\mathcal{C}_P\) by taking

- objects \(\text{obj} \mathcal{C}_P = P\)

- morphisms \(\mathcal{C}_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)

(where 1 is some fixed one-element set and \(\emptyset\) is the empty set)
Example: each pre-order determines a category

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- morphisms \(C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)
- identity morphisms and composition are uniquely determined (why?)
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- identity morphisms and composition are uniquely determined (why?)

E.g. when \((P, \sqsubseteq)\) has just one element 0

\[
C_P = \begin{array}{ccc}
0 & \text{id}_0 \\
\end{array}
\]

one object, one morphism
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(\mathcal{C}_P\) by taking

- objects \(\text{obj } \mathcal{C}_P = P\)
- morphisms \(\mathcal{C}_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)
- identity morphisms and composition are uniquely determined (why?)

E.g. when \((P, \sqsubseteq)\) has just two elements \(0 \sqsubseteq 1\)

\[
\mathcal{C}_P = \begin{array}{c}
\text{id}_0 \\
0 \\
\text{id}_1 \\
\end{array}
\begin{array}{c}
\circlearrowleft \\
\rightarrow \\
\circlearrowright \\
\end{array}
\begin{array}{c}
0 \\
\rightarrow \\
1 \\
\end{array}
\]

two objects, one non-identity morphism
Example: each pre-order determines a category

Given a pre-ordered set \( (P, \sqsubseteq) \), we get a category \( C_P \) by taking

- objects \( \text{obj } C_P = P \)
- morphisms \( C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases} \)
- identity morphisms and composition are uniquely determined (why?)

Example of a finite category that does not arise from a pre-ordered set:

\[
\begin{array}{ccc}
\text{id}_0 & 0 & \text{id}_1 \\
\circ & \circ & \circ \\
\text{two objects, two non-identity morphisms}
\end{array}
\]
Example: each monoid determines a category

Given a monoid \((M, \cdot, e)\), we get a category \(\mathbf{C}_M\) by taking

- **objects:** \(\text{obj } \mathbf{C}_M = 1 = \{0\}\) (one-element set)
- **morphisms:** \(\mathbf{C}_M(0, 0) = M\)
- **identity morphism:** \(\text{id}_0 = e \in M = \mathbf{C}_M(0, 0)\)
- **composition of** \(f \in \mathbf{C}_M(0, 0)\) **and** \(g \in \mathbf{C}_M(0, 0)\) **is** \(g \cdot f \in M = \mathbf{C}_M(0, 0)\)
Definition of isomorphism

Let \( \mathbf{C} \) be a category. A \( \mathbf{C} \)-morphism \( f : X \rightarrow Y \) is an isomorphism if there is some \( g : Y \rightarrow X \) for which

\[
\begin{align*}
  &X \xrightarrow{f} Y \\
  &\downarrow{g} \quad \downarrow{\text{id}_Y} \\
  &X \xrightarrow{\text{id}_X} X
\end{align*}
\]

is a commutative diagram.
Definition of isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

- Such a $g$ is uniquely determined by $f$ (why?) and we write $f^{-1}$ for it.
- Given $X, Y \in \mathbf{C}$, if such an $f$ exists, we say the objects $X$ and $Y$ are isomorphic in $\mathbf{C}$ and write $X \cong Y$.

(There may be many different $f$ that witness the fact that $X$ and $Y$ are isomorphic.)
**Theorem.** A function $f \in \text{Set}(X,Y)$ is an isomorphism in the category $\text{Set}$ iff $f$ is a bijection, that is

- **injective:** $\forall x, x' \in X, f \cdot x = f \cdot x' \Rightarrow x = x'$
- **surjective:** $\forall y \in Y, \exists x \in X, f \cdot x = y$

**Proof...**
**Theorem.** A function \( f \in \text{Set}(X, Y) \) is an isomorphism in the category \( \text{Set} \) iff \( f \) is a bijection, that is

- injective: \( \forall x, x' \in X, \, f(x) = f(x') \Rightarrow x = x' \)
- surjective: \( \forall y \in Y, \exists x \in X, \, f(x) = y \)

**Proof**...

**Theorem.** A monoid morphism \( f \in \text{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \) is an isomorphism in the category \( \text{Mon} \) iff \( f \in \text{Set}(M_1, M_2) \) is a bijection.

**Proof**...
Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.
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**Theorem.** A monotone function $f \in \text{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category **Poset** iff $f \in \text{Set}(P_1, P_2)$ is a surjective function satisfying

- **reflective:** $\forall x, x' \in P_1, f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'$

**Proof...**
Define **Poset** to be the category whose objects are posets = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

**Theorem.** A monotone function \( f \in \text{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \) is an isomorphism in the category **Poset** iff \( f \in \text{Set}(P_1, P_2) \) is a surjective function satisfying

\[
\text{reflective: } \forall x, x' \in P_1, \quad f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'
\]

**Proof...**

(Why does this characterisation not work for **Preord**?)
We have seen that
\[(M_1, \cdot_1, e_1) \cong (M_2, \cdot_2, e_2)\] in \textbf{Mon} \iff \[M_1 \cong M_2\] in \textbf{Set}.

However,
\[(P_1, \sqsubseteq_1) \cong (P_2, \sqsubseteq_2)\] in \textbf{Preord} \nless\[P_1 \cong P_2\] in \textbf{Set}.
We have seen that
\((M_1, \cdot_1, e_1) \cong (M_2, \cdot_2, e_2)\) in \textbf{Mon} \iff \(M_1 \cong M_2\) in \textbf{Set}.

However,
\((P_1, \sqsubseteq_1) \cong (P_2, \sqsubseteq_2)\) in \textbf{Preord} \not\iff P_1 \cong P_2\) in \textbf{Set}.

For example, consider
\[
P_1 = P_2 = \{0, 1\} \text{ a two-element set}
\]
\[
\sqsubseteq_1 = \{(0, 0), (1, 1)\}
\]
\[
\sqsubseteq_2 = \{(0, 0), (0, 1)(1, 1)\}
\]
for which we have \((P_1, \sqsubseteq_1) \not\cong (P_2, \sqsubseteq_2)\) \(\text{(why?)}\)

\[\text{Ex. Sh. 1, qn 1(b)}\]