Lecture 2
Recall

A category $\mathbf{C}$ is specified by

- a set $\text{obj } \mathbf{C}$ whose elements are called $\mathbf{C}$-objects
- for each $X, Y \in \text{obj } \mathbf{C}$, a set $\mathbf{C}(X, Y)$ whose elements are called $\mathbf{C}$-morphisms from $X$ to $Y$
- a function assigning to each $X \in \text{obj } \mathbf{C}$ an element $\text{id}_X \in \mathbf{C}(X, X)$ called the identity morphism for the $\mathbf{C}$-object $X$
- a function assigning to each $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ (where $X, Y, Z \in \text{obj } \mathbf{C}$) an element $g \circ f \in \mathbf{C}(X, Z)$ called the composition of $\mathbf{C}$-morphisms $f$ and $g$ and satisfying associativity and unity properties.
Example: category of pre-orders, $\textbf{Preord}$

objects are sets $P$ equipped with a pre-order $\sqsubseteq$ — a binary relation on $P$ that is
reflexive: $\forall x \in P, \ x \sqsubseteq x$
transitive: $\forall x, y, z \in P, \ x \sqsubseteq y \land y \sqsubseteq z \Rightarrow x \sqsubseteq z$

A partial order is a pre-order that is also
anti-symmetric: $\forall x, y \in P, \ x \sqsubseteq y \land y \sqsubseteq x \Rightarrow x = y$
Example: category of pre-orders, $\text{Preord}$

- objects are sets $P$ equipped with a pre-order $\preceq$

- morphisms: $\text{Preord}((P_1, \preceq_1), (P_2, \preceq_2)) \triangleq \{ f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone} \}$

\[ \forall x, x' \in P_1, \ x \preceq_1 x' \Rightarrow f x \preceq_2 f x' \]
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- Objects are sets $P$ equipped with a pre-order $\preceq$
- Morphisms: $\text{Preord}((P_1, \preceq_1), (P_2, \preceq_2)) \triangleq \{f \in \text{Set}(P_1, P_2) \mid f \text{ is monotone}\}$
- Identities and composition: as for $\text{Set}$

Q: why is this well-defined?
A: because the set of monotone functions contains identity functions and is closed under composition.
Example: category of pre-orders, $\mathbf{Preord}$

- objects are sets $P$ equipped with a pre-order $\sqsubseteq$
- morphisms: $\mathbf{Preord}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \triangleq \{f \in \mathbf{Set}(P_1, P_2) \mid f$ is monotone$\}$
- identities and composition: as for $\mathbf{Set}$

Pre- and partial orders are relevant to the denotational semantics of programming languages (among other things).
Example: category of monoids, \textbf{Mon}

\begin{itemize}
  \item objects are monoids \((M, \cdot, e)\) — set \(M\) equipped with a binary operation \(\cdot: M \times M \rightarrow M\) which is
  \item associative \(\forall x, y, z \in M, x \cdot (y \cdot z) = (x \cdot y) \cdot z\)
  \item has \(e\) as its unit \(\forall x \in M, e \cdot x = x = x \cdot e\)
\end{itemize}

CS-relevant example of a monoid: \((\text{List} \Sigma, @, \text{nil})\) where

\begin{align*}
\text{List} \Sigma & = \text{set of finite lists of elements of set } \Sigma \\
@ & = \text{list concatenation} \\
\text{nil} @ \ell & = \ell \\
(a :: \ell) @ \ell' & = a :: (\ell @ \ell') \\
\text{nil} & = \text{empty list}
\end{align*}
Example: category of monoids, \textbf{Mon}

- objects are monoids \((M, \cdot, e)\)

- morphisms: \(\textbf{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \textbf{Set}(M_1, M_2) \mid f e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (fx) \cdot_2 (fy) \}\)

It’s common to denote a monoid \((M, \cdot, e)\) just by its underlying set \(M\), leaving \_\cdot\_ and \(e\) implicit (hence the same notation gets used for different instances of monoid operations).
Example:
category of monoids, \( \textbf{Mon} \)

- objects are monoids \((M, \cdot, e)\)
- morphisms: \( \text{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \text{Set}(M_1, M_2) \mid f e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (fx) \cdot_2 (fy) \} \)
- identities and composition: as for \( \text{Set} \)

Q: why is this well-defined?
A: because the set of functions that are monoid morphisms contains identity functions and is closed under composition.
Example: category of monoids, $\text{Mon}$

- objects are monoids $(M, \cdot, e)$
- morphisms: $\text{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2)) \triangleq \{ f \in \text{Set}(M_1, M_2) \mid f e_1 = e_2 \land \forall x, y \in M_1, f(x \cdot_1 y) = (f x) \cdot_2 (f y) \}$
- identities and composition: as for $\text{Set}$

Monoids are relevant to automata theory (among other things).
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(\mathbf{C}_P\) by taking

- objects \(\text{obj } \mathbf{C}_P = P\)
- morphisms \(\mathbf{C}_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)

(where 1 is some fixed one-element set and \(\emptyset\) is the empty set)
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(\mathbf{C}_P\) by taking

- objects \(\text{obj} \mathbf{C}_P = P\)
- morphisms \(\mathbf{C}_P(x, y) \triangleq \begin{cases} 
1 & \text{if } x \sqsubseteq y \\
\emptyset & \text{if } x \not\sqsubseteq y
\end{cases}\)

identity morphisms and composition are uniquely determined (why?)
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- identity morphisms and composition are uniquely determined (why?)

E.g. when \((P, \sqsubseteq)\) has just one element \(0\)

\[
\mathbf{C}_P = \begin{array}{c}
0 \\
\circ \text{id}_0 \\
\text{one object, one morphism}
\end{array}
\]
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(C_P\) by taking

- objects \(\text{obj } C_P = P\)
- morphisms \(C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \not\sqsubseteq y \end{cases}\)
- identity morphisms and composition are uniquely determined (why?)

E.g. when \((P, \sqsubseteq)\) has just two elements \(0 \sqsubseteq 1\)

\[
C_P = \begin{array}{c}
\text{id}_0 & 0 & \longrightarrow & 1 & \text{id}_1 \\
\end{array}
\]

two objects, one non-identity morphism
Example: each pre-order determines a category

Given a pre-ordered set \((P, \sqsubseteq)\), we get a category \(C_P\) by taking

- **objects** \(\text{obj } C_P = P\)
- **morphisms** \(C_P(x, y) \triangleq \begin{cases} 1 & \text{if } x \sqsubseteq y \\ \emptyset & \text{if } x \nsubseteq y \end{cases}\)
- **identity morphisms** and composition are uniquely determined (why?)

Example of a finite category that does not arise from a pre-ordered set:

\[
\begin{array}{ccc}
id_0 & \circ & 0 & \circ & 1 & \circ & \text{id}_1 \\
| & \nleq & | & \nleq & | & \nleq & \\
\text{two objects, two non-identity morphisms} & & & & & &
\end{array}
\]
Example: each monoid determines a category

Given a monoid \((M, \cdot, e)\), we get a category \(\mathcal{C}_M\) by taking

- objects: \(\text{obj } \mathcal{C}_M = 1 = \{0\}\) (one-element set)
- morphisms: \(\mathcal{C}_M(0,0) = M\)
- identity morphism: \(\text{id}_0 = e \in M = \mathcal{C}_M(0,0)\)
- composition of \(f \in \mathcal{C}_M(0,0)\) and \(g \in \mathcal{C}_M(0,0)\) is \(g \cdot f \in M = \mathcal{C}_M(0,0)\)
Definition of isomorphism

Let $\mathcal{C}$ be a category. A $\mathcal{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ for which

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\text{id}_X} & & \downarrow{\text{id}_Y} \\
X & \xrightarrow{g} & Y
\end{array}
\]

is a commutative diagram.
Definition of isomorphism

Let $\mathbf{C}$ be a category. A $\mathbf{C}$-morphism $f : X \to Y$ is an isomorphism if there is some $g : Y \to X$ with $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.

- Such a $g$ is uniquely determined by $f$ (why?) and we write $f^{-1}$ for it.
- Given $X, Y \in \mathbf{C}$, if such an $f$ exists, we say the objects $X$ and $Y$ are isomorphic in $\mathbf{C}$ and write $X \cong Y$ (There may be many different $f$ that witness the fact that $X$ and $Y$ are isomorphic.)
Theorem. A function \( f \in \text{Set}(X,Y) \) is an isomorphism in the category \( \text{Set} \) iff \( f \) is a bijection, that is

- injective: \( \forall x, x' \in X, \ f x = f x' \Rightarrow x = x' \)
- surjective: \( \forall y \in Y, \exists x \in X, \ f x = y \)

Proof...
**Theorem.** A function $f \in \text{Set}(X, Y)$ is an isomorphism in the category $\text{Set}$ iff $f$ is a bijection, that is

- injective: $\forall x, x' \in X, f x = f x' \Rightarrow x = x'$
- surjective: $\forall y \in Y, \exists x \in X, f x = y$

**Proof...**

**Theorem.** A monoid morphism $f \in \text{Mon}((M_1, \cdot_1, e_1), (M_2, \cdot_2, e_2))$ is an isomorphism in the category $\text{Mon}$ iff $f \in \text{Set}(M_1, M_2)$ is a bijection.

**Proof...**
Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.
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**Theorem.** A monotone function $f \in \text{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2))$ is an isomorphism in the category **Poset** iff $f \in \text{Set}(P_1, P_2)$ is a surjective function satisfying

\begin{align*}
\text{reflective: } \forall x, x' \in P_1, \quad f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x'
\end{align*}

Proof...
Define **Poset** to be the category whose objects are **posets** = pre-ordered sets for which the pre-order is anti-symmetric, but is otherwise defined like the category **Preord** of pre-ordered sets.

**Theorem.** A monotone function
\[ f \in \text{Poset}((P_1, \sqsubseteq_1), (P_2, \sqsubseteq_2)) \]
is an isomorphism in the category **Poset** iff \( f \in \text{Set}(P_1, P_2) \) is a surjective function satisfying

\[ \forall x, x' \in P_1, \quad f x \sqsubseteq_2 f x' \Rightarrow x \sqsubseteq_1 x' \]

**Proof...**

(Why does this characterisation not work for **Preord**?)