1. [8/40 marks] Let $C$ be a category with binary products. Given a $C$-object $X$, the diagonal morphism $\delta_X \in C(X, X \times X)$ and the twist morphism $\tau_X \in C(X \times X, X \times X)$ are defined by:

$$\delta_X \triangleq (\text{id}_X, \text{id}_X) \quad \tau_X \triangleq \langle \pi_2, \pi_1 \rangle$$

(a) For all $C$-objects $X$ and $Y$ and all morphisms $f, g \in C(X, Y)$, show that

$$\tau_Y \circ (g \times f) = (f \times g) \circ \tau_X$$

(where $f \times g \in C(X \times X, Y \times Y)$ is the product of morphisms introduced in Ex. Sh. 2, question 1b).

(b) For each $f \in C(X, Y)$, show that $\delta_Y \circ f = (f \times f) \circ \delta_X$

(c) Show that $\tau_X \circ \delta_X = \delta_X$.

(d) Show that $\tau_X \circ \tau_X = \text{id}_{X \times X}$.

2. [10/40 marks] Let $C$ be a category. Given $C$-objects $X$ and $Y$ and morphisms $f, g \in C(X, Y)$, an equalizer for $f$ and $g$ is by definition a $C$-object $E$ and a morphism $m \in C(E, X)$ such that

- $f \circ m = g \circ m \in C(E, Y)$ and
- for all $C$-objects $Z$ and morphisms $h \in C(Z, X)$, if $f \circ h = g \circ h \in C(Z, Y)$, then there exists a unique morphism $k \in C(Z, E)$ satisfying $m \circ k = h$.

![Equalizer Diagram](image)

(a) Show that every equalizer is a monomorphism (see Ex. Sh. 1, question 4).

(b) Suppose that $f \in C(X, Y)$ is a split monomorphism, that is, there is a morphism $g \in C(Y, X)$ with $g \circ f = \text{id}_X$ (see Ex. Sh. 1, question 4). Show that $f : X \to Y$ is the equalizer of some pair of morphisms.

(c) Suppose the diagram

![Equalizer Diagram](image)

commutes in $C$ (i.e. $m \circ u = v \circ e$), that $m$ is an equalizer and that $e$ is an epimorphism (see Ex. Sh. 1, question 5). Show that there is a unique morphism $k \in C(V, E)$ such that $m \circ k = v$ and $k \circ e = u$. 
(d) Show that the category \textbf{Set} of sets and functions possesses equalizers for all parallel pairs of morphisms and that every monomorphism is an equalizer. Is every monomorphism in \textbf{Set} a split monomorphism?

3. [10/40 marks] Let \(X\) be an object of a category \(\mathbf{C}\). The slice category \(\mathbf{C} / X\) is defined by:

- The objects of \(\mathbf{C} / X\) are pairs \((A, p)\) where \(A \in \text{obj}\mathbf{C}\) and \(p \in \mathbf{C}(A, X)\).
- Given two such objects \((A, p)\) and \((B, q)\), a morphism \(f : (A, p) \to (B, q)\) in \(\mathbf{C} / X\) is a \(\mathbf{C}\)-morphism \(f \in \mathbf{C}(A, B)\) such that \(q \circ f = p\)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
p \downarrow & & \downarrow q \\
X & & \\
\end{array}
\]

- Composition and identities in \(\mathbf{C} / X\) are given by those in \(\mathbf{C}\).

(a) Show that \(\mathbf{C} / X\) always has a terminal object.
(b) Show that if \(\mathbf{C}\) has an initial object and binary coproducts, then so does \(\mathbf{C} / X\).
(c) When \(\mathbf{C} = \textbf{Set}\), the category of sets and functions, show that \(\textbf{Set} / X\) has binary products. [Hint: given \((A, p), (B, q) \in \text{obj}\textbf{Set} / X\), consider a suitable subset of \(\{(a, b) \mid a \in A \land b \in B\}\).]

4. [4/40 marks] Let \(\mathbf{C} = \textbf{Set}^{\text{op}}\) be the opposite category of the category \(\textbf{Set}\) of sets and functions.

(a) State, without proof, what is the product in \(\mathbf{C}\) of two objects \(X\) and \(Y\).
(b) Show by example that there are objects \(X\) and \(Y\) in \(\mathbf{C}\) for which there is no exponential and hence that \(\mathbf{C}\) is not a cartesian closed category.

5. [8/40 marks] Let \(\mathbf{C}\) be a cartesian closed category. Writing \(X \rightarrow Y\) for the exponential \(Y^X\) of two objects in \(\mathbf{C}\), define \(P(X, Y)\) to be the \(\mathbf{C}\)-object ((\(X \rightarrow Y\)) \rightarrow X) \rightarrow X.

(a) By giving a suitable simply typed lambda calculus term in the internal language of \(\mathbf{C}\), or otherwise, show that for any \(\mathbf{C}\)-object \(X\), there is a morphism \(p_X : 1 \rightarrow P(X, X)\) in \(\mathbf{C}\).
(b) When \(\mathbf{C} = \textbf{Set}\), show that for any sets \(X\) and \(Y\) (including the case where one or other of them is empty), there is always some morphism \(1 \rightarrow P(X, Y)\).
(c) Give an example of a cartesian closed category \(\mathbf{C}\) containing objects \(X\) and \(Y\) for which there is no morphism \(1 \rightarrow P(X, Y)\). [Hint: recall the example on page 63 in Lecture 6.]
(d) Call a term \(t\) of the simply typed lambda calculus \textit{pure} if it does not contain any constant symbols. Explain why part (c) implies that there is no pure term \(t\) such that \(\vdash t : ((G \to G') \to G) \to G\) holds, where \(G\) and \(G'\) are distinct ground types.