1. Show that for any objects $X$ and $Y$ in a cartesian closed category $C$, there are functions

$$f \in C(X,Y) \mapsto f^\top \in C(1,Y^X)$$
$$g \in C(1,Y^X) \mapsto \overline{g} \in C(X,Y)$$

that give a bijection between the set $C(X,Y)$ of $C$-morphisms from $X$ to $Y$ and the set $C(1,Y^X)$ of $C$-morphisms from the terminal object $1$ to the exponential $Y^X$. [Hint: use the isomorphism (7) from Exercise Sheet 2, question 2.]

2. Show that for any objects $X$ and $Y$ in a cartesian closed category $C$, the morphism $\text{app} : Y^X \times X \to Y$ satisfies $\text{cur}(\text{app}) = \text{id}_{Y^X}$. [Hint: recall from equation (4) on Exercise Sheet 2 that $\text{id}_{Y^X} \times \text{id}_X = \text{id}_{Y^X \times X}$.]

3. Suppose $f : Y \times X \to Z$ and $g : W \to Y$ are morphisms in a cartesian closed category $C$. Prove that

$$\text{cur}(f \circ (g \times \text{id}_X)) = (\text{cur} f) \circ g \in C(W, Z^X) \quad (1)$$

[Hint: use Exercise Sheet 2, question 1c.]

4. Let $C$ be a cartesian closed category. For each $C$-object $X$ and $C$-morphism $f : Y \to Z$, define

$$f^X \triangleq \text{cur}(Y^X \times X \xrightarrow{\text{app}} Y \xrightarrow{f} Z) \in C(Y^X, Z^X) \quad (2)$$

(a) Prove that $(\text{id}_Y)^X = \text{id}_{Y^X}$.

(b) Given $f \in C(Y \times X, Z)$ and $g \in C(Z, W)$, prove that

$$\text{cur}(g \circ f) = g^X \circ \text{cur} f \in C(Y, W^X) \quad (3)$$

(c) Deduce that if $u \in C(Y, Z)$ and $v \in C(Z, W)$, then $(v \circ u)^X = v^X \circ u^X \in C(Y^X, W^X)$. [Hint: for part (4a) use question 2; for part (4b) use Exercise Sheet 2, question 1c.]

5. Let $C$ be a cartesian closed category. For each $C$-object $X$ and $C$-morphism $f : Y \to Z$, define

$$X^f \triangleq \text{cur}(X^Z \times Y \xrightarrow{\text{id}_X \times f} X^Z \times Z \xrightarrow{\text{app}} X) \in C(X^Z, X^Y) \quad (4)$$

(a) Prove that $X^{\text{id}_Y} = \text{id}_{X^Y}$.

(b) Given $g \in C(W, X)$ and $f \in C(Y \times X, Z)$, prove that

$$\text{cur}(f \circ (\text{id}_Y \times g)) = Z^g \circ \text{cur} f \in C(Y, Z^W) \quad (5)$$

(c) Deduce that if $u \in C(Y, Z)$ and $v \in C(Z, W)$, then $X^{(v \circ u)} = X^u \circ X^v \in C(X^W, X^Y)$. [Hint: for part (5a) use question 2; for part (5b) use Exercise Sheet 2, question 1c.]
6. Let \( C \) be a cartesian closed category in which every pair of objects \( X \) and \( Y \) possesses a binary coproduct \( X \xleftarrow{\text{inl}_X} X + Y \xrightarrow{\text{inr}_Y} Y \). For all objects \( X, Y, Z \in C \) construct an isomorphism \( (Y + Z) \times X \cong (Y \times X) + (Z \times X) \). [Hint: you may find it helpful to use some of the properties from question 4.]

7. Using the natural deduction rules for Intuitionistic Propositional Logic (given in Lecture 6), give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.

   (a) \( \circ, \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi \)
   (b) \( \circ, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi \)
   (c) \( \circ, ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi \vdash \varphi \Rightarrow \psi \)

8. (a) Given simple types \( A, B, C \), give terms \( s \) and \( t \) of the Simply Typed Lambda Calculus that satisfy the following typing and \( \beta\eta \)-equality judgements:

\[
\begin{align*}
\circ, x : (A \times B) \rightarrow C & \vdash s : A \rightarrow (B \rightarrow C) & (6) \\
\circ, y : A \rightarrow (B \rightarrow C) & \vdash t : (A \times B) \rightarrow C & (7) \\
\circ, x : (A \times B) \rightarrow C & \vdash t[s/y] =_{\beta\eta} x : (A \times B) \rightarrow C & (8) \\
\circ, y : A \rightarrow (B \rightarrow C) & \vdash s[t/x] =_{\beta\eta} y : A \rightarrow (B \rightarrow C) & (9)
\end{align*}
\]

(b) Explain why question (8a) implies that for any three objects \( X, Y \) and \( Z \) in a cartesian closed category \( C \), there are morphisms

\[
\begin{align*}
f : Z^{(X \times Y)} & \rightarrow (Z^Y)^X & (10) \\
g : (Z^Y)^X & \rightarrow Z^{(X \times Y)} & (11)
\end{align*}
\]

that give an isomorphism \( Z^{(X \times Y)} \cong (Z^Y)^X \) in \( C \).

9. Make up and solve a question like question 8 ending with an isomorphism \( X^1 \cong X \) for any object \( X \) in a cartesian closed category \( C \) (with terminal object 1).