Exercise Problems 9–12: Information Theory

Exercise 9

Y and Z are two continuous random variables.

Y has an exponential probability density distribution $p(x)$ over $x \in [0, \infty]$: $p(x) = e^{-x}$.
Note that

$$\int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1\,.$$

Z has a uniform probability density distribution: $p(x) = 1/\alpha$ for $x \in [0, \alpha]$, else $p(x) = 0$.

Calculate the differential entropies $h(Y)$ and $h(Z)$ for these two continuous random variables, and find the value of α for which these differential entropies are the same. Sketch these distributions.

Exercise 10

(a) What does it mean for a function to be "self-Fourier?" Name three functions which are of importance in information theory and that have the self-Fourier property, and in each case mention a topic or a theorem exploiting it.

(b) Show that the set of all Gabor wavelets is closed under convolution, *i.e.* that the convolution of any two Gabor wavelets is just another Gabor wavelet. [HINT: This property relates to the fact that these wavelets are also closed under multiplication, and that they are also self-Fourier. You may address this question for just 1D wavelets if you wish.]

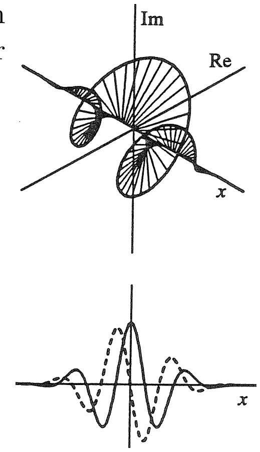
(c) Show that the family of sinc functions used in the Nyquist Sampling Theorem,

$$\mathrm{sinc}(x) = \frac{\sin(\lambda x)}{\lambda x}$$

is closed under convolution. Show further that when two different sinc functions are convolved, the result is simply whichever one of them had the lower frequency, *i.e.* the smaller λ.

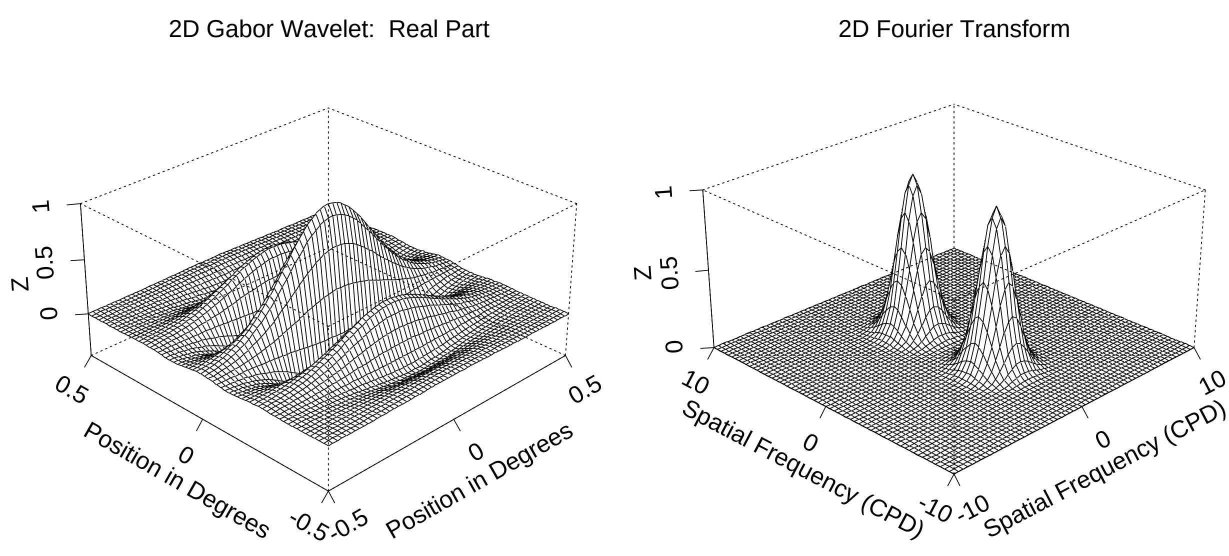
Exercise 11

(a) An important class of complex-valued functions for encoding information with maximal resolution simultaneously in the frequency domain and the signal domain are Gabor wavelets. Using an expression for their functional form, explain:



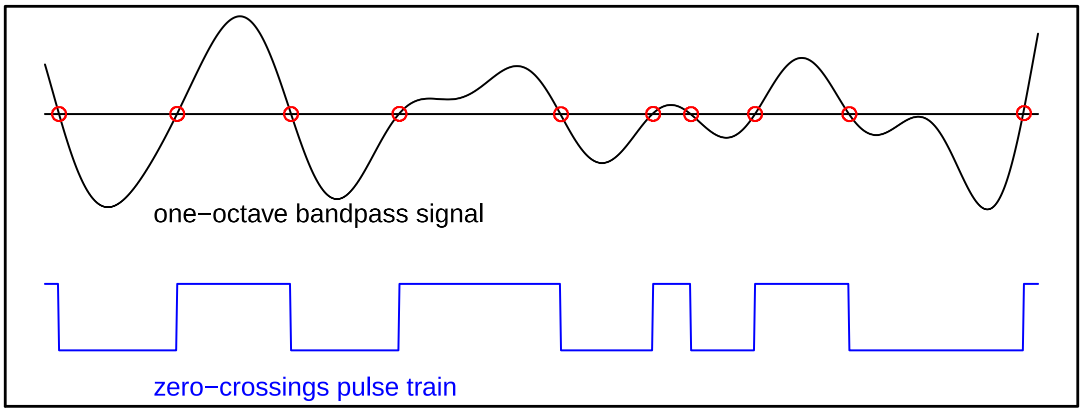
1. their spiral helical trajectory as phasors, shown here with projections of their real and imaginary parts;
2. the Uncertainty Principle under which they are optimal;
3. the spaces they occupy in the Information Diagram;
4. some of their uses in pattern encoding and recognition.

(b) Explain why the real-part of a 2D Gabor wavelet has a 2D Fourier transform with two peaks, not just one, as shown in the right panel of the Figure below.



Exercise 12

(a) Explain Logan's Theorem about the richness of the zero-crossings in signals bandlimited strictly to one octave, such as illustrated in the figure below. Name one intended application, and at least one algorithmic difficulty, of Logan's Theorem.



Next, consider *instead* an amplitude-modulated signal such as $f(t) = [1 + a(t)]\, c(t)$, where $c(t)$ is a pure sinusoidal carrier wave and its modulating function is anything $[1 + a(t)] > 0$. What does Logan's Theorem say about the information contained in the zero-crossings of $f(t)$?

(b) Compare and contrast the compression strategies deployed in the JPEG and JPEG-2000 protocols. Include these topics: the underlying transforms used; their computational efficiency and ease of implementation; artefacts introduced in lossy mode; typical compression factors; and their relative performance when used to achieve severe compression rates.

(c) Define the Kolmogorov algorithmic complexity K of a string of data, and say whether or not it is computable. What relationship is to be expected between the Kolmogorov complexity K and the Shannon entropy H for a given set of data? Give a reasonable estimate of K for a fractal, and explain why it is reasonable. Discuss the following concepts in Kolmogorov's theory of pattern complexity: how writing a program that generates a pattern is a way of compressing it, and executing such a program decompresses it; Kolmogorov incompressibility, and patterns that are their own shortest possible description.