Exercise Problems 5–8: Information Theory

Exercise 5

(a) An error-correcting (7/4) Hamming code combines four data bits b_3, b_5, b_6, b_7 with three error-correcting bits: $b_1 = b_3 \oplus b_5 \oplus b_7$, $b_2 = b_3 \oplus b_6 \oplus b_7$, and $b_4 = b_5 \oplus b_6 \oplus b_7$. The 7-bit block is then sent through a noisy channel, which corrupts one of the seven bits. The following incorrect bit pattern is received:

b_1	b_2	b_3	b_4	b_5	b_6	b_7
1	1	0	1	0	0	0

Evaluate three syndromes that can be derived upon reception of this corrupted 7-bit block: $s_1 = b_1 \oplus b_3 \oplus b_5 \oplus b_7$, $s_2 = b_2 \oplus b_3 \oplus b_6 \oplus b_7$, $s_4 = b_4 \oplus b_5 \oplus b_6 \oplus b_7$, and provide the corrected 7-bit block that was the original input to this noisy channel.

(b) Consider a binary symmetric channel with error probability p that any bit may be flipped. Two possible error-correcting coding schemes are available: Hamming, or simple repetition.

(i) Without any error-correcting coding scheme in place, state all the conditions that would maximise the channel capacity. Include conditions on the error probability p and also on the probability distribution of the binary source input symbols.

(ii) If a (7/4) Hamming code is used to deliver error correction for up to one flipped bit in any block of seven bits, provide an expression for the residual error probability P_e that such a scheme would fail.

(iii) If repetition were used to try to achieve error correction by repeating every message an odd number of times $N = 2m+1$, for some integer m followed by majority voting, provide an expression for the residual error probability P_e that the repetition scheme would fail.

Exercise 6

(a) What class of continuous signals has the greatest possible entropy for a given variance (*i.e.* power level)? What probability density function describes the excursions taken by such signals from their mean value?

(b) What does the Fourier power spectrum of this class of signals look like?

(c) Consider a noisy continuous communication channel of bandwidth $W = 1$ MHz, which is perturbed by additive white Gaussian noise whose total spectral power is $N_0 W = 1$. Continuous signals are transmitted across such a channel, with average transmitted power $P = 1{,}000$. Give a numerical estimate for the *channel capacity*, in bits per second, of this noisy channel. Then, for a channel having the same bandwidth W but whose signal-to-noise ratio $\frac{P}{N_0 W}$ is four times better, repeat your numerical estimate of capacity in bits per second.

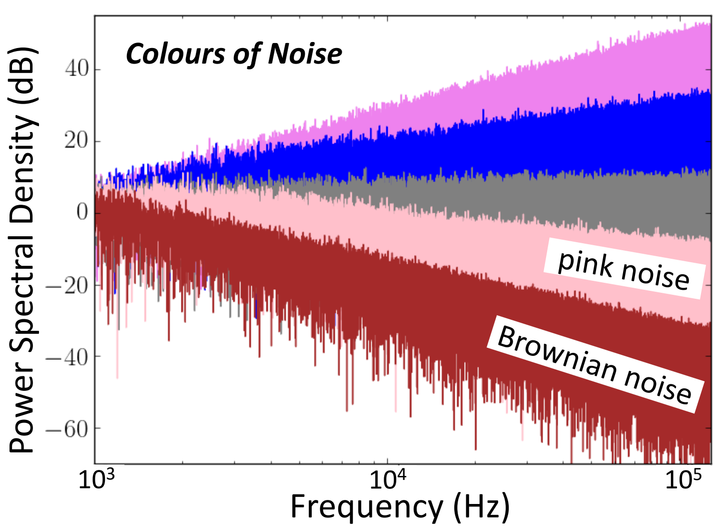
(d) Suppose that for such a continuous channel with added white Gaussian noise, the ratio of signal power to noise power is given as **30 decibels**, and the frequency bandwidth W of this channel is 10 MHz. Roughly what is the information capacity C of this channel, in bits/second?

(e) With no constraints on the parameters of such a channel, is there any limit to its capacity if you increase its signal-to-noise ratio $\dfrac{P}{N_0 W}$ without limit? If so, what is that limit?

(f) Is there any limit to the capacity of such a channel if you can increase its spectral bandwidth W (in Hertz) without limit, while not changing N_0 or P? If so, what is that limit?

Exercise 7

Shannon's *Noisy Channel Coding Theorem* showed how the capacity C of a continuous communication channel is limited by added white Gaussian noise; but other colours of noise are available. Among the "power-law" noise profiles shown in the figure as a function of frequency ω, Brownian noise has power that attenuates as $(\frac{\omega}{\omega_0})^{-2}$, and pink noise as $(\frac{\omega}{\omega_0})^{-1}$, above some minimum ω_0.



Consider three channels suffering from either white, pink, or Brownian noise. At frequency $\omega = \omega_0$ all three channels have the same signal-to-noise ratio $\text{SNR}(\omega_0)$ and it remains at this level for the white channel, but at higher frequencies ω it improves as $(\frac{\omega}{\omega_0})$ for the pink channel and as $(\frac{\omega}{\omega_0})^2$ for the Brownian channel. Show that across any frequency band $[\omega_1, \omega_2]$ $(\omega_0 < \omega_1 < \omega_2)$ the Brownian and the pink noise channels have higher capacity than the white noise channel, and show that as frequency grows large the Brownian channel capacity approaches **twice** that of the pink channel.

Exercise 8

(a) An inner product space V is spanned by an orthonormal system of vectors $\{e_1, e_2, \ldots, e_n\}$ so that $\forall i \neq j$ the inner product $\langle e_i, e_j \rangle = 0$, but every e_i is a unit vector so that $\langle e_i, e_i \rangle = 1$. We wish to represent a data set consisting of vectors $u \in \text{span}\{e_1, e_2, \ldots, e_n\}$ in this space as a linear combination of the orthonormal vectors: $u = \sum_{i=1}^{n} a_i e_i$. Derive how the coefficients a_i can be determined for any vector u, and comment on the computational advantage of representing the data in an orthonormal system.

(b) An inner product space containing complex functions $f(x)$ and $g(x)$ is spanned by a set of orthonormal basis functions $\{e_i\}$. Complex coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ therefore exist such that $f(x) = \sum_i \alpha_i e_i(x)$ and $g(x) = \sum_i \beta_i e_i(x)$.

Show that the inner product $\langle f, g \rangle = \sum_i \alpha_i \overline{\beta_i}$.

(c) Consider a noiseless analog communication channel whose bandwidth is 10,000 Hertz. A signal of duration 1 second is received over such a channel. We wish to represent this continuous signal exactly, at all points in its one-second duration, using just a finite list of real numbers obtained by sampling the values of the signal at discrete, periodic points in time. What is the length of the shortest list of such discrete samples required in order to guarantee that we capture all of the information in the signal and can recover it exactly from this list of samples?

(d) Name, define algebraically, and sketch a plot of the function you would need to use in order to recover completely the continuous signal transmitted, using just such a finite list of discrete periodic samples of it.

(e) Explain why smoothing a signal, by low-pass filtering it *before* sampling it, can prevent aliasing. Explain aliasing by a picture in the Fourier domain, and also show in the picture how smoothing solves the problem. What would be the most effective low-pass filter to use for this purpose? Draw its spectral sensitivity.

(f) If a continuous signal $f(t)$ is *modulated* by multiplying it with a complex exponential wave $\exp(i\omega t)$ whose frequency is ω, what happens to the Fourier spectrum of the signal?

Name a very important practical application of this principle, and explain why modulation is a useful operation. How can the original Fourier spectrum later be recovered?