

FoCS Lecture 2

Recursion and Complexity

14th October 2019

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Expression Evaluation

$$E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n \rightarrow v$$

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Focus on *expressions*;
ignore *side-effects* for now.

This discipline of separating expression
from effects is often known as
functional programming

We will return to side effects later in the
course to make useful programs!

Expression Evaluation

$$E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n \rightarrow v$$

```
# let rec power x n =  
  if n = 1 then x  
  else if even n then  
    power (x *. x) (n / 2)  
  else  
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```

Expression Evaluation

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```

power(2, 12) \Rightarrow

power(4, 6) \Rightarrow

power(16, 3) \Rightarrow

16 \times power(256, 1) \Rightarrow

16 \times 256 \Rightarrow

4096

Summing first n integers

```
# let rec nsum n =  
  if n = 0 then  
    0  
  else  
    n + nsum (n - 1)
```

$$\begin{aligned}nsum\ 3 &\Rightarrow 3 + (nsum\ 2) \\ &\Rightarrow 3 + (2 + (nsum\ 1)) \\ &\Rightarrow 3 + (2 + (1 + nsum\ 0)) \\ &\Rightarrow 3 + (2 + (1 + 0))\end{aligned}$$

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Nothing can progress
until the final expression
is calculated!

Summing first n integers

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```

Intermediate results are stored in the program *stack* which is usually of limited size.

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 $\Rightarrow 3 + (2 + (nsum\ 1))$
 $\Rightarrow 3 + (2 + (1 + nsum\ 0))$
 $\Rightarrow 3 + (2 + (1 + 0))$

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Iteratively summing

```
# let rec summing n total =  
  if n = 0 then  
    total  
  else  
    summing (n - 1) (n + total)
```

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# let rec nsum n =  
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Iteratively summing

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# let rec summing n total =  
  if n = 0 then  
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```

$summing\ 3\ 0 \Rightarrow summing\ 2\ 3$
 $\Rightarrow summing\ 1\ 5$
 $\Rightarrow summing\ 0\ 6$
 $\Rightarrow 6$

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Iteratively summing

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# let rec summing n total =  
  if n = 0 then  
    total  
  else  
    summing (n - 1) (n + total)
```

summing 3 0 ⇒ *summing 2 3*
⇒ *summing 1 5*
⇒ *summing 0 6*
⇒ 6

Extra argument `total`
acts as the *accumulator*
to keep track explicitly
instead of using the stack

Algorithms like this are
known as *iterative* or
tail recursive

Recursion vs iteration

- Why two terms *iterative* and *tail recursive*?
 - “Iterative” normally refers to a loop: e.g. coded using `while`.
 - “Tail-recursion” involves the recursive function call being the last thing that expression does.
- Tail-recursion is efficient only if the compiler detects it.
 - Mainly it saves space, though iterative code can run faster.
- Do not make programs iterative unless you determine the gain is significant.

How can we
analyse our
programs for
efficiency?

Silly summing first n integers

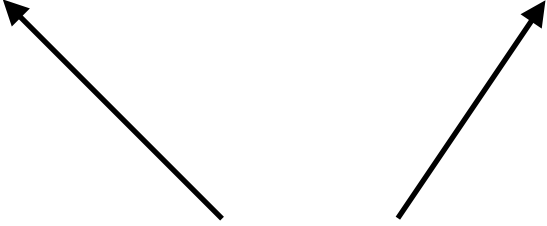
```
# let rec sillySum n =  
  if n = 0 then  
    0  
  else  
    n + (sillySum (n-1) + sillySum (n-1)) / 2
```



Recursively calls itself
twice for every invocation

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twice for every invocation

Should **assign** the result to
a local variable to prevent
evaluating it twice

```
# let x = 2.0 in  
  let y = Float.pow x 20.0 in  
  y *. (x /. y)
```

Asymptotic complexity refers to how program costs grow with increasing inputs

Usually space or time, with the latter usually being larger than the former.

Question: if we double our processing power, how much does our computation capability increase?

Time Complexity

<i>complexity</i>	<i>1 second</i>	<i>1 minute</i>	<i>1 hour</i>	gain
n	1000	60,000	3,600,000	$\times 60$
$n \lg n$	140	4,893	200,000	$\times 41$
n^2	31	244	1,897	$\times 8$
n^3	10	39	153	$\times 4$
2^n	9	15	21	$+6$

complexity = milliseconds of runtime given an input of size n

Comparing Algorithms with $O(n)$

Formally, define $f(n) = O(g(n))$
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Intuitively, consider the *most significant term*
and ignore constant or smaller factors

E.g. simplify $3n^2 + 34n + 433 \rightarrow n^2$

Facts about O notation

$O(2g(n))$ is the same as $O(g(n))$

$O(\log_{10} n)$ is the same as $O(\ln n)$

$O(n^2 + 50n + 36)$ is the same as $O(n^2)$

$O(n^2)$ is contained in $O(n^3)$

$O(2^n)$ is contained in $O(3^n)$

$O(\log n)$ is contained in $O(\sqrt{n})$

Common complexity classes

$O(1)$	<i>constant</i>
$O(\log n)$	<i>logarithmic</i>
$O(n)$	<i>linear</i>
$O(n \log n)$	<i>quasi-linear</i>
$O(n^2)$	<i>quadratic</i>
$O(n^3)$	<i>cubic</i>
$O(a^n)$	<i>exponential (for fixed a)</i>

Sample costs in O-notation

<i>function</i>	<i>time</i>	<i>space</i>
<code>npower, nsum</code>	$O(n)$	$O(n)$
<code>summing</code>	$O(n)$	$O(1)$
$n(n + 1)/2$	$O(1)$	$O(1)$
<code>power</code>	$O(\log n)$	$O(\log n)$
<code>stupSum</code>	$O(2^n)$	$O(n)$

Simple recurrence relations

$T(n)$: a cost we want to bound using O notation

Typical *base case*: $T(1) = 1$

Some *recurrences*:

$$T(n + 1) = T(n) + 1 \quad O(n)$$

$$T(n + 1) = T(n) + n \quad O(n^2)$$


$$T(n) = T(n/2) + 1 \quad O(\log n)$$

$$T(n) = 2T(n/2) + n \quad O(n \log n)$$

Mapping this to OCaml

```
# let rec nsum n =  
  if n = 0 then  
    0  
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```

Given $(n+1)$, does a
constant amount of
work



Then calls itself
with n



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Therefore, recurrence relations are:

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Calls itself
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Calls nsum which
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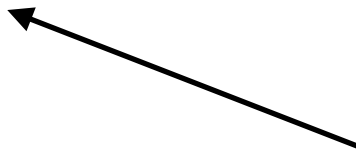
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Therefore, recurrence relations are:

$$\begin{aligned} T(0) &= 1 \\ T(n + 1) &= T(n) + n \end{aligned} \quad O(n^2)$$

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← Calls itself
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← Always divides
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